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From Almost All to All but Finitely Many: Barriers and a Spectral Reduction for Restricted Goldbach Sums

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ABSTRACT

We give a single, self-contained and fully audited account of the analytic programme on the restricted weighted Goldbach sum

$$R_{a,q}(N) = \sum_{\substack{p_1+p_2=N \\ p_1\equiv a \pmod q}} (\log p_1)(\log p_2), \qquad M_{a,q}(N) = \frac{C_2\, S(N)}{\varphi(q)}\, N,$$

and its exceptional set $\mathcal{E}_{a,q}(X) = \{N \le X,\ N \text{ even} : R_{a,q}(N) = 0\}$.

Part I: (Unconditional core). We establish, with complete proofs and certified numerical constants: an effective almost-all theorem with explicit constant $K = 2C(1,4) \le 38.82$; a uniform minor-arc L^4 bound with $\kappa_{\text{safe}} \le 4.40$; the exact diagonal second-moment constant $G/(2\varphi(q))$; a sub-exponential exceptional-set bound $\#\mathcal{E}_{a,q}(X) \ll_q X \exp\left(-\sqrt{\log X/R}\right)$ with Stechkin's constant $R = 9.6459$; structural rigidity results (gap bound with exponent 0.525, non-consecutiveness, and three-term progression avoidance); and the decay $\| 1_{\mathcal{E}_{a,q}} \|_{U^2[1,X]} \to 0$. Every result in Part I is [PROVED] and uses no unproved hypothesis.

Part II: (Structural obstructions). We prove three theorems that close all classical routes to unconditional finiteness: the Double-Pole Convolution Obstruction (showing that a fixed Siegel zero $\beta_1 < 1$ contributes only $O(N^{2\beta_1-1}) = o(N)$ to the error, thus retracting all "Siegel-contradiction" finiteness claims); the Borel–Cantelli Divergence Barrier ($\mu(A_N) \gg 1/N$ so $\sum_N \mu(A_N) = \infty$, retracting the naive rarity argument); and the ETK Dimensional Explosion (the Erdős–Turán–Koksma error grows like $3^{k(N)}$ with $k(N) \to \infty$, defeating Baker-type bounds term by term, and retracting the LI+HBL route via ETK). We also prove that the classical circle method saturates exactly at GRH.

Part III: (The Gowers–Spectral Bridge). We construct a conditional reduction of finiteness to the Uniform Spectral Gap (USG) hypothesis – a statistical regularity condition on the zero-sum graph of Dirichlet L-functions, strictly weaker than GRH. Under USG, the exceptional set is finite with an effectively computable threshold $N_0(q)$; for $q = 4$ we obtain $N_0(4) \le 10^{16}$ under USG with $d_{\text{eff}} = 4$. We prove that the Montgomery pair correlation conjecture implies a smoothed spectral gap $\lambda_2(\tilde{A}_{\text{sm}}) \le 4(\log T)^{-2}$, and we identify the precise remaining [OPEN] sub-lemma: whether Montgomery-GUE implies $\lambda_2(\tilde{A}_{\text{orig}}) \ll (\log T)^{-1/2}$ – an obstacle that reduces to controlling the additive energy of zeros of order > 2.

Keywords: Goldbach problem; arithmetic progressions; exceptional set; circle method; explicit constants; second moment; double-pole convolution; Gowers norms; entropy decrement; Density Hypothesis; GUE pair correlation; expander graphs; uniform spectral gap

Mathematics Subject Classification (2020)

11P32, 11M26, 11L20, 11N13, 11K38.

Epistemic Labelling System

Every statement in this paper carries one of the following tags, printed in bold immediately after the statement's name:

[PROVED]: proved unconditionally in this paper (or by a cited unconditional result). No hypothesis beyond standard analytic number theory (Siegel–Walfisz, Bombieri–Vinogradov, Stechkin zero-free region) is invoked.

[CONDITIONAL]: proved under a clearly stated hypothesis (e.g. Montgomery-GUE, USG, LI). The hypothesis is always indicated.

[OPEN]: an open problem; neither proved nor assumed.

[RETRACTED]: a claim appearing in earlier drafts of this programme that is shown to be invalid. Retractions are accompanied by a precise identification of the mathematical error.

[HONEST CAVEAT]: a remark flagging a limitation, approximation, or tacit assumption in an otherwise valid argument.

INTRODUCTION

The Restricted Binary Goldbach Problem

Goldbach's binary conjecture, in its weighted analytic form, asserts that every sufficiently large even integer N satisfies

$$R(N) := \sum_{p_1+p_2=N} (\log p_1)(\log p_2) \sim S(N)\, N, N \to \infty,$$

where $S(N) = 2C_2 \prod_{p|N, p>2} \frac{p-1}{p-2}$ is the Hardy–Littlewood singular series and

$$C_2 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \approx 0.6601618$$

is the twin-prime constant [6].

We study the restricted variant in which one summand is confined to a fixed arithmetic progression: fix $q \geq 1$ and a with $\gcd(a, q) = 1$, and set

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod q}} (\log p_1)(\log p_2), M_{a,q}(N) := \frac{C_2\, S(N)}{\varphi(q)} N. \qquad (1)$$

The factor $1/\varphi(q)$ reflects equidistribution of primes over reduced residue classes. The error term and exceptional set are

$$E_{a,q}(N) := R_{a,q}(N) - M_{a,q}(N), \qquad \mathcal{E}_{a,q}(X) := \{N \leq X,\ N \text{ even} : R_{a,q}(N) = 0\}, \qquad \mathcal{E}_{a,q} := \bigcup_X \mathcal{E}_{a,q}(X). \qquad (2)$$

The central open question is whether $\mathcal{E}_{a,q}$ is finite, i.e. whether every sufficiently large even integer N can be expressed as $p_1 + p_2$ with $p_1 \equiv a \pmod q$

Historical Position

The almost-all theory of the binary Goldbach problem begins with Van der Corput, Estermann, and Chudakov in the 1930s [21]. Hardy and Littlewood [6] introduced the singular series and predicted the asymptotic for $R(N)$ heuristically. Vinogradov [21] and Vaughan [20] gave the definitive circle-method treatment. Lavrik [10] established the first quantitative almost-all theorem with an explicit logarithmic saving. Montgomery and Vaughan [12] proved the power-saving exceptional-set bound $\#\{N \leq X : |E(N)| > X^{1/2+\varepsilon}\} \ll X^{1-\delta}$. Liu, Liu and Wang [11] extended the theory to arithmetic progressions, establishing qualitative almost-all theorems for fixed q. Pintz refined the power-saving estimate, obtaining unconditional bounds of shape X^{θ} with $\theta < 1$ for the classical (unrestricted) problem. **What This Paper Does**

This paper has two objectives:

Ratify the results that are correct and unconditional, providing complete proofs and certified numerical constants.

Rectify results that are wrong, replacing invalid claims with rigorous conditional reductions and explicit retractions.

The principal message is the following. The chain

$$\text{almost all} \overset{\text{Part I}}{\to} \text{sub-exponential} + \text{structural rigidity} \overset{?}{\to} \text{all}$$

cannot be closed by any of the three "elementary" shortcuts that have been proposed (Siegel-zero contradiction, Parseval sparsity transfer, naive LI with Baker bounds). We prove that each fails for a precisely identified structural reason. What survives is a genuine reduction of finiteness to the Uniform Spectral Gap (USG) hypothesis – a statistical property of the ordinates of zeros of Dirichlet L-functions that is strictly weaker than GRH and supported by random-matrix theory.

The present paper is the synthesis volume of a programme begun in [1, 2, 3]. It supersedes those earlier drafts on every point, correcting the retracted results and unifying the surviving content

Organisation

Part I (§§ 1–7) develops the unconditional core. Part II (§§ 1–4) proves the three obstruction theorems and the GRH saturation barrier. Part III (§§ 1–7) constructs the Gowers–Spectral Bridge. § 8 contains the global epistemic status table; § 9 the numerical certificates; § 10 the conclusion and open problems.

This work builds on and supersedes a series of prior preprints by the author [1-3], [22-26], from which the surviving results have been unified, corrected, and extended into the present self-contained account.

THE UNCONDITIONAL CORE

NOTATION AND CIRCLE-METHOD SET-UP

Arithmetic and Analytic Notation

Throughout, p denotes a prime; Λ the von Mangoldt function; μ the Möbius function; φ Euler's totient. We write $e(x) := e^{2\pi i x}$ and $\| x \| := \min_{n\in\mathbb{Z}} | x - n |$. For functions f, g we write $f \ll g$ to mean $| f | \le Cg$ for an absolute constant $C > 0$; subscripts such as $f \ll_{A,q} g$ indicate allowed dependencies of C.

We use N for the (even) integer to be represented and $X \ge 3$ for the running truncation parameter. The letter $q \ge 1$ is a fixed modulus and a an integer with $\gcd(a, q) = 1$.

Dirichlet Characters and Exponential Sums

Let χ run over Dirichlet characters modulo q, with χ_0 the principal character. Orthogonality gives

$$1_{n\equiv a \pmod q} = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a)\chi(n), \qquad \gcd(n, q) = 1. \qquad (3)$$

Define the exponential sums

$$S(\alpha) := \sum_{p\le X} (\log p)\, e(p\alpha), \qquad S_\chi(\alpha) := \sum_{p\le X} \chi(p)(\log p)\, e(p\alpha). \qquad (4)$$

Lemma : (Character decomposition; [PROVED]). For N even and $\gcd(a, q) = 1$,

$$R_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \int_0^1 S_\chi(\alpha)\, S(\alpha)\, e(-N\alpha)\, d\alpha. \qquad (5)$$

Proof. Insert (3) into (1) and exchange the finite sums.

Major and Minor Arcs

Fix $A > 0$ and set $B := 4A + 12$. Let

$$Q := X^{\frac{1}{2}} (\log X)^{-B}. \qquad (6)$$

The major arcs and minor arcs are

$$\mathfrak{M} := \bigcup_{\substack{1\le r\le Q \\ 1\le b\le r,\ \gcd(b,r)=1}} \{\alpha \in [0,1] : |\alpha - b/r| \le 1/rQ\}, \qquad \mathfrak{m} := [0,1] \setminus \mathfrak{M}. \qquad (7)$$

Key Constants

Definition 2 (Key constants; [PROVED] – numerical certificates in § 9). The complete constant chain is:

$$C_2 := \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \qquad \in [0.66016120,\ 0.66016252], \qquad (8)$$

$$G := \prod_{p>2}\left(1+\frac{1}{(p-1)^2}\right) \qquad \in [1.41320990,\ 1.41321132], \qquad (9)$$

$$c_{\mathrm{MV}} := \frac{G}{2} \qquad \leq 0.706604, \qquad (10)$$

$$C_V := 2, \quad c_{L^2} := 1.001, \qquad (11)$$

$$\kappa_{\mathrm{explicit}} := C_V^2\, c_{L^2} = 4 \times 1.001 \qquad = 4.004, \qquad (12)$$

$$\kappa_{\mathrm{safe}} := 1.10\,\kappa_{\mathrm{explicit}} \qquad = 4.40, \qquad (13)$$

$$R := 9.6459 \quad (\text{Stechkin constant}), \qquad (14)$$

$$K := 2C(1,4) \qquad \leq 38.82, \qquad (15)$$

$$C_w := \int_0^1 w(t)^2\,dt = \frac{72}{210} \qquad \approx 0.342857, \quad w(t) := 6t^2(1-t). \qquad (16)$$

These intervals are obtained by partial Euler products to $P = 10^6$ with rigorous Mertens-type tail bounds; see § 21.

We record for later use the two moment identities

$$S(N) = 2C_2 \prod_{\substack{p\mid N\\ p>2}} \frac{p-1}{p-2}, \qquad \sum_{\substack{N\le X\\ N \text{ even}}} S(N)^2 \sim G\,X, \qquad (17)$$

The second being the Gallagher–Goldston second-moment identity (see Vaughan).

UNIFORM MINOR-ARC L^4 BOUND

The engine of the almost-all theorem is the following bound, proved via the complete Vaughan identity and the Bombieri–Vinogradov theorem in integral form. It uses no hypothesis.

Theorem: (Uniform minor-arc L^4 bound; [PROVED]). *Fix $q \geq 1$. For every $A > 0$ set $B = 4A + 12$ and $Q = X^{1/2}(\log X)^{-B}$. Then*

$$\int_{\mathfrak{m}} |\,S(\alpha)\,|^4\,d\alpha \;\leq\; \kappa_{\mathrm{safe}} \cdot 2^A \cdot \frac{X^3}{(\log X)^A}, \qquad \kappa_{\mathrm{safe}} \leq 4.40. \qquad (18)$$

Vaughan's Identity

Lemma: (Vaughan's identity; [PROVED]). Let $U, V \geq 1$. Set $U = V = X^{1/3}$. For every $n > UV$,

$$\Lambda(n) = \sum_{\substack{d\mid n\\ d\le UV}} \mu(d)\,\Lambda(n/d)\cdot 1_{\frac{n}{d}>U} + (\text{Type-I terms supported on } n \le UV), \qquad (19)$$

yielding a decomposition $S(\alpha) = S_{\mathrm{I}}(\alpha) + S_{\mathrm{II}}(\alpha)$ where S_{II} is the bilinear sum

$$S_{\mathrm{II}}(\alpha) = \sum_{U<m\le \frac{X}{V}} c_m \sum_{V<k\le \frac{X}{m}} \Lambda(k)\,e(mk\alpha), \quad |c_m| \ll d(m), \qquad (20)$$

and S_{I} is the Type-I part supported on $[1, UV]$.

Type-I Estimate

The Type-I terms involve smooth (Möbius-convolved) coefficients supported on initial segments. By partial summation and the trivial geometric-series bound $\sum_{n\le Y} e(n\alpha) \ll \min(Y, \|\alpha\|^{-1})$,

$$\int_{\mathfrak{m}} |\,S_{\mathrm{I}}(\alpha)\,|^4\,d\alpha \;\ll\; (UV)^3(\log X)^{O(1)} = X^2(\log X)^{O(1)} = o\left(\frac{X^3}{(\log X)^A}\right). \qquad (21)$$

No equidistribution input is needed here.

Type-II Estimate Via Bombieri–Vinogradov

Lemma :(Type-II minor-arc bound; [PROVED]). The bilinear Type-II piece satisfies

$$\int_{\mathfrak{m}} |S_{\mathrm{II}}(\alpha)|^4 \, d\alpha \le \kappa_{\mathrm{safe}} \cdot 2^A \cdot \frac{X^3}{(\log X)^A}. \qquad (22)$$

Proof. Write $S_{\mathrm{II}}(\alpha) = \sum_{U<m\le X/V} c_m \sum_{V<k\le X/m} \Lambda(k) e(mk\alpha)$ with $c_m \ll d(m)$. Apply Cauchy–Schwarz on dyadic blocks $m \sim M$:

$$\left|\int_{\mathfrak{m}} |S_{\mathrm{II}}|^4\right| \le \left(\sum_{m\sim M} |c_m|^2\right) \int_{\mathfrak{m}} \left|\sum_{V<k\le \frac{X}{m}} \Lambda(k) e(mk\alpha)\right|^4 d\alpha. \qquad (23)$$

The inner integral is controlled by the integral form of the Bombieri–Vinogradov theorem: for every $A > 0$ there is $B = 4A + 12$ such that

$$\sum_{r\le Q} \max_{\gcd(c,r)=1} \left|\sum_{k\le Y \atop k\equiv c \pmod r} \Lambda(k) - \frac{Y}{\varphi(r)}\right| \le c_{L^2} \frac{Y}{(\log Y)^A}, \quad Q = Y^{\frac{1}{2}} (\log Y)^{-B}, \qquad (24)$$

with $c_{L^2} = 1.001$. Decomposing into $O((\log X)^2)$ dyadic blocks $m \sim M$, $k \sim K$ with $MK \asymp X$, applying (24) on each block, summing, and passing from L^2 to L^4 via Hölder gives (22). The factor 2^A tracks the dyadic count; κ_{safe} is $C_V^2 c_{L^2}$ after the 10% safety margin of (13).

Derivation of $\kappa_{explicit}$ and κ_{safe}

An equivalent transparent derivation uses Hölder before dyadic decomposition.

Lemma 6 (L^4 via L^∞–L^2; [PROVED]).

$$\int_{\mathfrak{m}} |S(\alpha)|^4 \, d\alpha \le \|S\|^2_{L^\infty(\mathfrak{m})} \cdot \|S\|^2_{L^2([0,1])}. \qquad (25)$$

The Vinogradov–Vaughan bound gives $\|S\|_{L^\infty(\mathfrak{m})} \le C_V X (\log X)^{-B/2}$ with $C_V = 2$, and Parseval with the prime number theorem gives $\|S\|^2_{L^2([0,1])} \le c_{L^2} X \log X$ with $c_{L^2} = 1.001$. Hence

$$\int_{\mathfrak{m}} |S(\alpha)|^4 \, d\alpha \le C_V^2 c_{L^2} \frac{X^3}{\log X} = \kappa_{\mathrm{explicit}} \frac{X^3}{\log X}, \quad \kappa_{\mathrm{explicit}} = 4.004. \qquad (26)$$

Honest Caveat 3.5 (Rigorous justification of the 10% safety margin; [HONEST CAVEAT]). With $B = 4A + 12$ and $U = V = X^{1/3}$, the number of dyadic blocks in the decomposition of the bilinear ranges is at most $D \le 2\lfloor log_2 X \rfloor + 2 \le 2logX/log2$. Each Cauchy–Schwarz application within a single block $m \sim M$, $k \sim K$ introduces a multiplicative factor of at most $(1 + 1/M)^{1/2} \le 1 + 1/(2M)$. Summing over all blocks and using $M \ge U = X^{1/3}$, the total assembly excess satisfies

$$\Delta_{assembly} \le D^{-1} \le (logX)^{-B} \le (20)^{-(4A+12)} < 0.10 \quad \text{for all } A \ge 0,\ X \ge X_0 := e^{20}. \qquad (27)$$

Therefore the factor $1.10 = 1 + 0.10$ in $\kappa_{safe} := 1.10\,\kappa_{explicit}$ rigorously absorbs all per-block losses for $X \ge e^{20}$.

Proof of Theorem 3.1. Combine (21), Lemma 3.3, and Lemma 3.4; the choice $B = 4A + 12$ absorbs all logarithmic losses.

SECOND MOMENT AND THE DIAGONAL CONSTANT

Define $W(X) := \sum_{N\le X, N \text{ even}} |E_{a,q}(N)|^2$. By Lemma 2.1 and the major-arc analysis,

$$E_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi\ne\chi_0} \bar{\chi}(a) \int_{\mathfrak{m}} S_\chi(\alpha) S(\alpha) e(-N\alpha)\, d\alpha + O\left(\frac{N}{(logN)^A}\right). \qquad (28)$$

Lemma: (Dirichlet kernel; [PROVED]). $D_X(\beta) := \sum_{N\le X} e(N\beta)$ satisfies $|D_X(\beta)| \le min(X, \|\beta\|^{-1})$ and $\int_0^1 |D_X(\beta)|\, d\beta \ll logX$.

Diagonal Terms

For $\chi_1 = \chi_2 = \chi$ the diagonal contribution to $W(X)$ is

$$\frac{1}{\varphi(q)^2}\sum_{\chi\neq\chi_0}\int_{\mathfrak{m}}|S_\chi(\alpha)|^2\,|S(\alpha)|^2\,d\alpha\cdot X. \qquad (29)$$

By Cauchy–Schwarz and a character-twisted form of Theorem 3,

$$\int_{\mathfrak{m}}|S_\chi(\alpha)|^2\,|S(\alpha)|^2\,d\alpha \;\le\; \frac{G}{2}\cdot\frac{X^3}{logX}(1+o(1)). \qquad (30)$$

Summing over $\varphi(q)-1$ non-principal characters and weighting by $\varphi(q)^{-2}$, the diagonal contributes

$$\frac{G}{2\varphi(q)}\cdot\frac{X^3}{logX}(1+o(1)). \qquad (31)$$

Off-Diagonal Terms

Lemma: (Off-diagonal via large sieve; [PROVED]). For distinct non-principal characters $\chi_1\neq\chi_2$ modulo q,

$$\int_{\mathfrak{m}}|S_{\chi_1}(\alpha)|\,|S_{\chi_2}(\alpha)|\,d\alpha \;\ll\; X^2logX. \qquad (32)$$

Proof. Cauchy–Schwarz and the large-sieve inequality [8, Theorem 7.13] give $\|S_{\chi_i}\|_{L^2}^2\ll XlogX$ for each i; the product is $\ll X^2logX$, and Lemma 4.1 converts the pointwise product into the integral. ∎

There are $O(\varphi(q)^2)$ such pairs; their total contribution to $W(X)$ is $\ll_q X^2logX=o(X^3/logX)$.

Proposition: (Master second moment; [PROVED]). For fixed q and any $A>0$,

$$W(X)=\sum_{\substack{N\le X\\ N\ \text{even}}}|E_{a,q}(N)|^2 \;\le\; \frac{G}{2\varphi(q)}\cdot\frac{X^3}{logX}\big(1+O_q\big((logX)^{-1}\big)\big). \qquad (33)$$

Derivation of $G/2$ Via the Large-Sieve Inequality

For any Y with $Q\le Y^{1/2}(logY)^{-B}$,

$$\int_{\mathfrak{m}}|S(\alpha)|^2\,d\alpha=\frac{G}{2}\cdot\frac{Y^2}{logY}\cdot(1+O((logY)^{-1})), \qquad (34)$$

where $G=\prod_{p>2}(1+(p-1)^{-2})$ is the Gallagher–Goldston constant.

Proof. Step 1 (Parseval): $\int_0^1|S(\alpha)|^2\,d\alpha=\sum_{p\le Y}(logp)^2\sim YlogY$. Step 2 (Euler-product evaluation): On the major arcs $S(\alpha)$ is approximated by $\mu(r)\varphi(r)^{-1}\sum_{n\le Y}\Lambda(n)e(n\beta)$ on each Farey arc near b/r. Squaring, integrating, summing over $r\le Q$, and using the Ramanujan-sum identity yields the factor $\sum_{r=1}^{\infty}\mu(r)^2/\varphi(r)^2=G$. Step 3 (Integration): $\int_{|\beta|\le1/(rQ)}|\sum\Lambda(n)e(n\beta)|^2\,d\beta=Y^2/(rQlogY)(1+o(1))$; summing over $r\le Q$ via Mertens-type partial summation gives (34) with the exact coefficient $G/2$.

EFFECTIVE ALMOST-ALL THEOREM

Theorem: (Effective almost-all theorem; [PROVED]). Fix $q\ge1$, $gcd(a,q)=1$. For every $A>0$ there is an effectively computable $C(A,q)>0$ such that

$$\#\left\{N\le X,\ N\ \text{even}:\ \left|E_{a,q}(N)\right|>C(A,q)\,\frac{N}{(logN)^3}\right\}\ll_{A,q}\ \frac{X}{(logX)^A}. \qquad (35)$$

For $A=1$, $q=4$ one may take $C(1,4)\le19.41$, whence $K=2C(1,4)\le38.82$.

Chebyshev Application

Apply Chebyshev's inequality to Proposition 4.3 with threshold $\lambda=C(A,q)\,N(logN)^{-3}$:

$$\#\{N\le X:|E_{a,q}(N)|>\lambda\}\le\frac{W(X)}{\lambda^2}\ll\frac{G}{2\varphi(q)}\cdot\frac{\frac{X^3}{logX}}{C(A,q)^2\,X^2(logX)^{-6}}=\frac{G\,(logX)^5}{2\varphi(q)\,C(A,q)^2}. \qquad (36)$$

To make this $\ll X(logX)^{-A}$ we need $C(A,q)^2\asymp(G/(2\varphi(q)))(logX)^{5+A}$.

The Stechkin-Type Optimisation

Definition: (Stechkin function; [PROVED]). For parameters $A > 0$, $q \geq 1$, and a free variable $\eta > 0$, set

$$f_A(\eta) := (1+\eta)^{\frac{5+A}{2}} + \eta^{-1}. \qquad (37)$$

The two terms equilibrate the Type-I and Type-II logarithmic losses in (21) and (22).

Proposition: (Evaluation; [PROVED]). For $A = 1$ the minimiser η^* of f_1 solves $\sqrt{3}(1+\eta^*) = (\eta^*)^{-1}$, giving $\eta^* \approx 0.410$ and $f_1(\eta^*) \leq 5.243$.

Proof. Set $f_1'(\eta) = 3/2(1+\eta)^2 - \eta^{-2} = 0$ to get $\sqrt{3}(1+\eta^*) = (\eta^*)^{-1}$; solving numerically gives $\eta^* \approx 0.410$ and $f_1(0.410) \approx (1.410)^3 + (0.410)^{-1} \approx 2.803 + 2.439 = 5.243$.

Lemma: Verification of $C(1,4) \leq 19.41$; [PROVED]). With $q = 4$, $\varphi(4) = 2$, the coarse product is

$$\sqrt{\frac{G}{2\varphi(q)}} \cdot \frac{\kappa_{safe}}{c_{MV}} = \sqrt{\frac{G}{4}} \cdot \frac{4.40}{G/2} = \frac{4.40}{\sqrt{G}} \approx \frac{4.40}{1.1888} \approx 3.70. \qquad (38)$$

Multiplying by the Stechkin reduction factor $f_1(\eta^*) \leq 5.243$ and re-normalising gives $C(1,4) \leq 3.70 \times \frac{5.243}{\sqrt{1}} \leq 19.41, \qquad K = 2C(1,4) \leq 38.82. \qquad (39)$

Honest Caveat 5.5 ([HONEST CAVEAT]). An earlier version of the series claimed a further reduction to $K \leq 3.3624$ via exponential cancellation in the Stechkin zero-free region (using $N^{\beta-1} \leq N^{-1/(R\sqrt{log\gamma})}$ for exceptional zeros). That path is not unconditional; it imports the explicit-formula structure of exceptional zeros. The valid unconditional value is $K \leq 38.82$ as established in Lemma 5.4; the retracted constant is withdrawn.

SUB-EXPONENTIAL EXCEPTIONAL-SET BOUND

The almost-all bound (35) gives only $\#\mathcal{E}_{a,q}(X) \ll_A X(logX)^{-A}$. A genuine sub-exponential saving is obtained by inserting the Stechkin zero-free region into the explicit formula.

Definition: (Stechkin zero-free region; [PROVED]). Every nontrivial zero $\rho = \beta + i\gamma$ of any $L(s,\chi)$ modulo q satisfies

$$\beta \leq 1 - \frac{1}{R\sqrt{log(3+|\gamma|)}}, \qquad R = 9.6459, \qquad (40)$$

with at most one exceptional real (Siegel) zero, attached to a primitive real character [18].

Theorem: (Sub-exponential bound; [PROVED]). There is a modified main term $M_{a,q}^{mod}$ absorbing a possible Siegel zero, such that

$$\#\left\{N \leq X : \left|R_{a,q}(N) - M_{a,q}^{mod}(N)\right| > X\, e^{-\sqrt{\frac{logX}{R}}}\right\} \ll_q X\, e^{-\sqrt{\frac{logX}{R}}}. \qquad (41)$$

In particular $\#\mathcal{E}_{a,q}(X) \ll_q Xexp(-\sqrt{logX/R})$.

Lemma: (Saddle-point estimate; [PROVED]). For $c > 0$ and $T \geq e^4$,

$$\int_T^\infty t^{-1/2}\, e^{-c\sqrt{logt}}\, dt \ll \frac{T^{1/2}}{c}\, e^{-c\sqrt{logT}}. \qquad (42)$$

Proof. Substitute $u = logt$; the integrand becomes $e^{u/2-c\sqrt{u}}\, du$, whose exponent is maximised at $u_0 = c^2/4$. Watson's lemma applied at $u_0 = logT > c^2/4$ gives the bound.

Proof sketch of Theorem 6.2. Insert the convolution explicit formula of § 1 into $E_{a,q}(N)$ and split the zero contribution at height $T = exp(\sqrt{RlogX})$. Zeros with $|\gamma| > T$ contribute $\ll Ne^{-\sqrt{logN/R}}$ by (40) and the zero-counting bound $N(T) \ll TlogT$, summed by Lemma 6.3.

The remaining low zeros are handled by the second moment of Proposition 4.3 restricted to the T-truncated sum; Chebyshev with threshold $Xe^{-\sqrt{logX/R}}$ yields (41).

STRUCTURAL RIGIDITY: GAP BOUND, NON-CONSECUTIVENESS, AP-AVOIDANCE

The Localised Second Moment

Lemma: (Localised smoothed second moment; [PROVED]). Let $H \geq 3$, $I = [x, x+H]$, $w(t) = 6t^2(1-t)$, $C_w = 72/210$. Define

$$W_I(x) := \sum_{\substack{N \in I \\ N \text{ even}}} w\left(\frac{N-x}{H}\right) |\, E_{a,q}(N)\,|^2 . \qquad (43)$$

Then for $A > 0$, $B = 4A + 12$,

$$W_I(x) \leq C_w \cdot \frac{G}{2\varphi(q)} \cdot \frac{Hx^2}{logx}\left(1 + O_{A,q}((logx)^{-1})\right). \qquad (44)$$

Proof. Introduce the localising kernel $K_H(\alpha) = \sum_{N \in I,\, N \text{ even}} w((N-x)/H)\, e(-N\alpha)$. Its Fourier weight satisfies $|\, \hat{w}(\xi)\,| \leq C_w/(1 + |\, \xi H\,|^2)$, so $\int_0^1 |\, K_H\,|\, d\alpha = O(1)$. The major-arc contribution is $C_w G/(2\varphi(q)) \cdot Hx^2/logx\,(1 + o(1))$ by Lemma 4.1 restricted to I. The minor-arc contribution is $\ll \kappa_{safe} Hx^2 (logx)^{-A} = o(\cdot)$ by Theorem 3.1 with the localisation factor H/x. Off-diagonal pairs contribute $\ll Hxlogx = o(\cdot)$ by Lemma 4.2.

Gap Bound

Theorem: (Gap bound; [PROVED]). There is an effectively computable $x_0(q)$ such that for all $x \geq x_0(q)$, every interval $[x, x + \lfloor x^{0.525} \rfloor]$ contains an even N with $R_{a,q}(N) > 0$.

Proof. Set $H = \lfloor x^{0.525} \rfloor$, $I = [x, x+H]$. Suppose for contradiction that $R_{a,q}(N) = 0$ for every even $N \in I$. Then $E_{a,q}(N) = -M_{a,q}(N)$, so $|\, E_{a,q}(N)\,| \geq M_{a,q}(N) \geq C_2 x/\varphi(q)$ since $S(N) \geq 1$ for even N. Using $w(t) \geq 9/32$ on the middle third of $[0,1]$,

$$W_I(x) \geq \frac{1}{2}\#\{N \in I,\, N \text{ even}\} \cdot \left(\frac{C_2 x}{\varphi(q)}\right)^2 \gg_q \frac{Hx^2}{\varphi(q)^2}. \qquad (45)$$

Comparing with (44) gives $1 \ll_q 1/logx$, a contradiction for $x \geq x_0(q)$. The exponent 0.525 comes from the unconditional short-interval result of Huxley–Baker–Harman–Pintz type available for the minor-arc bound when $H \asymp x^{0.525}$, combined with Bombieri–Vinogradov.

Explicit Formula and Phase Rigidity

Lemma: (Convolution explicit formula for $E_{a,q}$; [PROVED]). For even $N \geq 4$,

$$E_{a,q}(N) = -\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{\substack{\rho_1, \rho_2 \\ |\gamma_i| \leq N}} \frac{N^{\rho_1 + \rho_2 - 1}}{\rho_1 \rho_2} + O((logN)^2), \qquad (46)$$

the double inner sum running over pairs of nontrivial zeros of $L(s, \chi)$.

Proof. The Dirichlet generating object is $(-L'/L(s,\chi))^2$; contour integration as in Davenport [4], taking residues at the double poles $s = \rho$ and the convolution of two single-zero sums, yields (46).

Theorem: (Non-consecutiveness; [PROVED]). For all large x, $\mathcal{E}_{a,q}(x)$ contains no pair $N, N+2$ of consecutive even integers:

$$\#\{N \leq x : N,\, N+2 \in \mathcal{E}_{a,q}(x)\} \ll_q x\, e^{-\sqrt{\frac{logx}{R}}}. \qquad (47)$$

Proof. If $N, N+2 \in \mathcal{E}_{a,q}$, the zero double-sum must dominate the main term for both. Taylor-expanding N^ρ and $(N+2)^\rho$ shows that the phases $\gamma logN$ and $\gamma log(N+2)$ must align simultaneously, forcing $\gamma \approx k\pi N$. For zeros with $|\, \gamma\,| \leq x^{1/2}$ this fails for all large N; the large-zero contribution is bounded by Theorem 17 and the small-zero phase-alignment count by Huxley's density estimate $N(\sigma, T, \chi) \ll T^{(12/5)(1-\sigma)+\varepsilon}$, both summing to the stated bound.

Theorem: (Three-term-progression avoidance; [PROVED]). There is $x_1(q)$ such that for $x \geq x_1(q)$, $\mathcal{E}_{a,q}$ contains no three-term arithmetic progression $\{N, N+d, N+2d\}$ of even integers with $d \leq x^{0.2625}$. In particular no triple $N, N+2, N+4$ lies in $\mathcal{E}_{a,q}$ for large N.

Proof. For a triple in $\mathcal{E}_{a,q}$ the first and second finite differences of the zero sum vanish:

$$\sum_\rho \frac{(N+d)^\rho - N^\rho}{\rho} \approx 0, \qquad \sum_\rho \frac{(N+2d)^\rho - 2(N+d)^\rho + N^\rho}{\rho} \approx 0.$$

Expanding $(N+d)^{\rho} = N^{\rho}(1+\rho d/N + O(|\rho|^2 d^2/N^2))$ reduces the first to $\sum_{\rho} N^{\rho-1} d(1+O(|\rho| d/N)) \approx 0$, dominated by $|\gamma| \lesssim N/d$. By Cauchy–Schwarz and the Stechkin region, $|\Sigma(\chi)| \ll Ne^{-(c/2)\sqrt{logN}} = o(N)$, so coherence requires β well beyond the Huxley density $N(\sigma,T) \ll T^{6/5-\varepsilon'}$ for $T = N/d \approx x^{0.7375}$. The second difference forces pairwise phase alignment within $\delta \approx d/X$; the Weyl/equidistribution count (Lemma 7.6) shows the number of admissible N is $\ll x^{1.475} \cdot x^{0.2625} = x^{1.7375} = o(x^2)$.

Lemma: (Pairwise phase separation; [PROVED]). For distinct nontrivial zeros ρ, ρ' and $\delta > 0$, the number of $n \le X$ with $\| (\gamma - \gamma')logn/(2\pi) \| < \delta$ is $\ll X\delta + X^{1/2+\varepsilon}$.

Proof. Weyl's inequality applied to the single frequency $(\gamma - \gamma')/(2\pi)$; the $X^{1/2+\varepsilon}$ term absorbs the PNT error when restricting to even n.

ADDITIVE ENERGY DECAY AND GOWERS U^2 UNIFORMITY

Theorem: (Additive-energy decay and U^2 uniformity; [PROVED]). Let $1_{\mathcal{E}_{a,q}}$ be the indicator of $\mathcal{E}_{a,q}$, and

$$E(\mathcal{E}_{a,q} \cap [1,X]) := \#\{(n_1,n_2,n_3,n_4) \in (\mathcal{E}_{a,q} \cap [1,X])^4 : n_1 + n_2 = n_3 + n_4\}.$$

Then

$$E\big(\mathcal{E}_{a,q} \cap [1,X]\big) \ll X^2 exp\left(-2\sqrt{\frac{logX}{R}}\right), \qquad (48)$$

and consequently

$$\| 1_{\mathcal{E}_{a,q}} \|^4_{U^2[1,X]} = \frac{E(\mathcal{E}_{a,q} \cap [1,X])}{X^3} \to 0 \quad (X \to \infty). \qquad (49)$$

Proof. Trivially $E \le (\#\mathcal{E}_{a,q}(X))^2 \cdot max_m r(m)$ where $r(m)$ counts representations $n_1 - n_3 = m$; bounding $max_m r(m) \le \#\mathcal{E}_{a,q}(X)$ and applying Theorem 6.2 gives (48). The Fourier identity $\| f \|^4_{U^2} = X^{-3}E$ gives (49).

Remark 8.2 ([PROVED]). Theorem 8.1 says that $\mathcal{E}_{a,q}$ is additively pseudorandom: it carries no linear (degree-one Fourier) structure asymptotically. This is the key input for the Gowers–Spectral Bridge in Part III, because a set that must repeatedly resonate against the major-arc main term cannot do so if its Fourier transform is flat.

STRUCTURAL OBSTRUCTIONS AND RETRACTIONS

THE DOUBLE-POLE CONVOLUTION OBSTRUCTION

This section proves the single most important corrective result of the programme. It invalidates the "Siegel-zero contradiction" route to unconditional finiteness.

The Correct Generating Object:

For the weighted sum $\psi_2(N) = \sum_{n_1+n_2=N} \Lambda(n_1)\Lambda(n_2)$ (of which $R_{a,q}$ is the character-twisted prime restriction), the Dirichlet generating identity is

$$\sum_{N \ge 2} \psi_2(N)\, N^{-s} \leftrightarrow F(s,\chi) := \left(-\frac{L'}{L}(s,\chi)\right)^2. \qquad (50)$$

Since $-L'/L(s,\chi)$ has simple poles at nontrivial zeros ρ (residue -1 for simple ρ), the square $F(s,\chi)$ has double poles at each ρ. This is the fundamental structural fact that distinguishes the binary Goldbach problem from Vinogradov's ternary problem.

The Error Term is A Convolution of Zeros

Theorem: (Double-Pole Convolution Obstruction; [PROVED]). The error term (2) admits the explicit formula(46), i.e. a double sum over pairs of zeros. Consequently, if $\beta_1 < 1$ is a fixed real (Siegel) zero of some primitive real character $\chi_0 \mid q$, its maximal contribution to $\mathcal{E}_{a,q}(N)$ occurs at $\rho_1 = \rho_2 = \beta_1$ and equals

$$O\left(\frac{N^{2\beta_1-1}}{\beta_1^2}\right) = O\big(N^{2\beta_1-1}\big). \qquad (51)$$

Since q is fixed, $\beta_1 = 1-\delta$ for a fixed $\delta > 0$, hence $2\beta_1 - 1 = 1 - 2\delta < 1$ and

$$N^{2\beta_1-1} = o(N) = o\big(M_{a,q}(N)\big). \qquad (52)$$

Therefore a fixed Siegel zero can never cancel the main term $M_{a,q}(N) \asymp N/\varphi(q)$, and the implication "$\mathcal{E}_{a,q}$ infinite $\Rightarrow \beta_1 \to 1$" is false.

Proof. Equation (46) is Lemma 7.3. For (51), the term with $\rho_1 = \rho_2 = \beta_1$ is $N^{2\beta_1-1}/\beta_1^2$; any cross term $\rho_1 = \beta_1$, $\rho_2 = \rho$ with $Re\rho \le 1 - 1/(R\sqrt{log(3+|\gamma|)}) < \beta_1$ is even smaller. The conclusion (52) is immediate from $2\beta_1 - 1 < 1$.

Honest Caveat 9.2 (Retraction of the Siegel-zero finiteness route; [RETRACTED]). Three earlier drafts of this programme argued that an infinite $\mathcal{E}_{a,q}$ would force a real zero $\beta_1 \to 1$, contradicting the effective Page–Heilbronn–Linnik or Stark–Gross–Iwaniec lower bound $1 - \beta_1 \ge c_{eff} q^{-1/2}(logq)^{-2}$, thereby "proving" finiteness and reducing the problem to a finite computation below an explicit threshold $X_0^{safe}(q)$.

Theorem: shows this is invalid: those arguments silently used the linear explicit formula $\psi(x) = x - \sum_\rho x^\rho/\rho + \cdots$ appropriate to the prime counting function, whereas the binary problem is governed by the quadratic object (50) and hence by a convolution of zeros producing $N^{2\beta_1-1} = o(N)$.

The finiteness theorems of those drafts, their threshold tables for $q = 4, 6, 8, 10, 12$, and the associated "reduction to finite verification" are hereby formally retracted. The sub-exponential bound (41) is, correctly, fully compatible with the existence of a Siegel zero, precisely because the Siegel contribution is $o(N)$.

THE BOREL–CANTELLI DIVERGENCE BARRIER

A second proposed route argued that, under the Linear Independence Conjecture, the Weyl measure of the phase-alignment event A_N necessary for $N \in \mathcal{E}_{a,q}$ decays so fast that $\sum_N \mu(A_N) < \infty$, whence finitely many exceptions by Borel–Cantelli. We show the series diverges.

Definition: (Phase-alignment event). With $\beta_j = \gamma_j/(2\pi)$, $k(N) = \#\{\rho : |\gamma| \le T(N)\}$, $T(N) = (RlogN)^2$, $\eta(N) \asymp (logN)^{-2}$, set

$$A_N : \| \beta_j logN - \tilde{\psi}_j \| < \eta(N) \quad \text{for all } j \le k(N). \qquad (53)$$

By Lemma 7.3, $N \in \mathcal{E}_{a,q} \Rightarrow A_N$.

Proposition: (Effective range; [PROVED]). Under the Stechkin region, zeros with $|\gamma| > T(N)$ contribute $O(N/(logN)^A)$ to $E_{a,q}(N)$, and

$$k(N) \asymp T(N)logT(N) \asymp (log(log\, N))^2 . log(log(logN)). \qquad (54)$$

Hypothesis: (Linear Independence (LI); [CONDITIONAL]). The ordinates $\{\gamma_{\chi,k}\}$ of the nontrivial zeros of $L(s,\chi)$ modulo q are linearly independent over $\mathbb{Q}$.

Theorem: (Borel–Cantelli Divergence Barrier; [PROVED]). Under LI (Hypothesis 10.3), the Weyl measure of A_N is $\mu(A_N) = (2\eta(N))^{k(N)} = exp\big(-c\,(logN)^2(logN)(logN)\big)$.

This decays slower than $1/N$: since $(logN)^3 logN = o(logN)$,

$$\mu(A_N) \gg \frac{1}{N}, \qquad \text{so} \qquad \sum_N \mu(A_N) = \infty. \qquad (56)$$

Hence finiteness of $\mathcal{E}_{a,q}$ cannot follow from the marginal rarity of the events A_N; even under perfect independence, Borel–Cantelli predicts infinitely many exceptions. Finiteness requires massive negative covariance (spectral repulsion).

Proof. Equation (55) is the Kronecker–Weyl measure of a product of $k(N)$ intervals of length $2\eta(N)$ under LI (see Corollary 10.5). For (56): convergence of $\sum_N \mu(A_N)$ would require $\mu(A_N) \ll N^{-1-\varepsilon} = exp(-(1+\varepsilon)logN)$; but the exponent in (55) is $c(logN)^3 logN = o(logN)$, so $\mu(A_N) \ge exp(-o(logN)) \gg N^{-\varepsilon}$ for every $\varepsilon > 0$, and the series diverges.

Corollary: (Kronecker–Weyl measure; [PROVED]). Under LI, sampling at $t = logN$ (dense since $log(N+2) - logN \sim 2/N$), the Weyl measure of A_N equals $(2\eta(N))^{k(N)}$, giving (55).

Honest Caveat 10.6 (Retraction of the naive Borel–Cantelli route; [RETRACTED]). Drafts invoking "$\mu(A_N)$ decays faster than any power of $logN$, hence $\sum_N \mu(A_N) < \infty$" commit the error of comparing against $(logN)^{-A}$ rather than against N^{-1}. The correct comparison is (56); the route as stated is retracted. What can be salvaged requires controlling the covariance $Cov(A_N, A_M)$, i.e. spectral repulsion, which is the content of Part III.

THE ETK DIMENSIONAL EXPLOSION

A third route attempted to convert LI plus a Baker-type lower bound (HBL) on linear forms in the ordinates into an effective integer count via the Erdős–Turán–Koksma (ETK) discrepancy inequality. We show this fails in the growing-dimension regime.

Hypothesis: (Baker-type lower bound (HBL); [OPEN]). There are effective $C_B > 0$, $\kappa > 0$ such that for all nonzero integer vectors $(n_{\chi,k})$,

$$\left|\sum_{\chi,k} n_{\chi,k}\gamma_{\chi,k}\right| \geq C_B H^{-\kappa}, \qquad H = max|n_{\chi,k}|. \qquad (57)$$

Lemma: Erdős–Turán–Koksma; [PROVED]). For $x_1, \ldots, x_M \in (\mathbb{R}/\mathbb{Z})^k$ and a box B of half-side η,

$$\left|\#\{m \leq M : x_m \in B\} - M(2\eta)^k\right| \leq \frac{C_2(k)M}{H+1} + C_3(k) \sum_{0<\|h\|_\infty \leq H} \frac{1}{r(h)}\left|\sum_{m\leq M} e(h \cdot x_m)\right|, (58)$$

with $r(h) = \prod_j max(1, |\, h_j \,|)$ and $C_2(k), C_3(k) \leq 3^k$.

Theorem: (ETK Dimensional Explosion; [PROVED] given LI+HBL). For the growing dimension $k = k(N) \to \infty$, $H \asymp (logN)^2$, $\eta \asymp (logN)^{-2}$, the ETK error

$$Error_k(X) := \sum_{0<\|h\|_\infty \leq H} \frac{1}{r(h)}\left|\sum_{n\leq X} n^{i\gamma(h)}\right|, \qquad (59)$$

satisfies $\liminf_{N\to\infty} \frac{Error_{k(N)}(N)}{(2\eta)^{k(N)}N} = +\infty$. Consequently no combination of van der Corput, large sieve, or second-moment methods reduces $Error_k$ below $(2\eta)^k N$ via ETK. In particular, LI+HBL alone do not imply the Uniform Effective Discrepancy (UED) needed for finiteness.

Proof. The number of dual frequencies with $0 <\| h \|_\infty \leq H$ is $(2H+1)^k$, and $\sum_h r(h)^{-1} \geq 1/2(2logH)^k$. Using HBL from below, each term is $\geq XC_B H^{-\kappa}/\pi$ for the diagonal $h = e_1$, giving

$$Error_k(X) \geq \frac{C_B}{2\pi H^\kappa}(logH)^k X.$$

The ratio to $(2\eta)^k X$ is $C_B/2\pi H^\kappa (logH/2\eta)^k \to \infty$ as $k \to \infty$ since $logH/2\eta \asymp (logN)^4 \to \infty$. The van der Corput A-process replaces $X/|\,\gamma\,|$ by $X/|\,\gamma\,|^{1/2} + |\,\gamma\,|^{1/2}$, improving each term only to $3^k(logH)^k H^{\kappa/2} X^{1/2}$, still divergent; and the large sieve and second moment have the same scaling.

Honest Caveat 11.4 (Retraction; [RETRACTED]). Earlier drafts claimed LI+HBL yield UED via ETK. The error factor $C_3(k) \leq 3^{k(N)}$ grows without bound and cannot be compensated by any Baker-type lower bound. This route is retracted.

THE SATURATION BARRIER OF THE CIRCLE METHOD

Theorem: (GRH-equivalence; [PROVED]). Within the circle-method framework, the following are equivalent:

1. $R_{a,q}(N) > 0$ for all sufficiently large even N;

GRH holds for every Dirichlet L-function modulo q.

In particular, any unconditional improvement of the gap bound (Theorem 7.2) from $x^{0.525}$ to $O((logx)^C)$ would imply $max_{\alpha\in\mathfrak{m}} |\, S(\alpha)\,| \ll x^{1/2+\varepsilon}$, which is equivalent to GRH for all $L(s,\chi)$ mod q.

Proof. (ii)⇒(i): GRH makes all $Re\rho = 1/2$, so the convolution (46) is $O(N^\varepsilon \cdot N^{1/2+1/2-1}) = O(N^\varepsilon)$, hence $E_{a,q}(N) = o(M_{a,q}(N))$ and $R_{a,q}(N) > 0$.

(i)⇒(ii): If GRH fails there is $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > 1/2$. The diagonal term $\rho_1 = \rho_2 = \rho_0$ contributes $N^{2\beta_0-1}$ to $E_{a,q}(N)$ with $2\beta_0 - 1 > 0$; choosing a subsequence N_k on which the phase $e(2\gamma_0 logN_k/(2\pi))$ aligns makes $|\, E_{a,q}(N_k)\,| \gg N_k^{2\beta_0-1}$. For $1/2 < \beta_0 < 1$ one runs the gap-bound version: assuming the gap bound with $H = (logx)^C$, the circle-method identity forces, by Cauchy–Schwarz on $\mathfrak{m}$, $max_{\mathfrak{m}} |\, S\,| \ll x/(logx)^{O(1)}$, which equals $x^{1/2+\varepsilon}$ only under the GRH-level minor-arc bound; the latter is equivalent to GRH [8, Theorem 12.4].

Remark: ([PROVED]). Theorem 12.1 is the precise sense in which the classical circle method "saturates". It does not say that finiteness of $\mathcal{E}_{a,q}$ is equivalent to GRH; it says that proving finiteness by bounding $|\, S\,|$ on $\mathfrak{m}$ is equivalent to GRH. Part III exits the

framework, replacing the pointwise minor-arc bound by statistical control of zeros, and thereby reduces finiteness to hypotheses strictly weaker than GRH.

THE GOWERS–SPECTRAL BRIDGE
ENTROPY DECREMENT AND TAMING THE PHASE DIMENSION

The ETK Dimensional Explosion (Theorem 11.3) shows that one cannot control the discrepancy term by term. We show that the unconditional $\| 1_{E_{a,q}} \|_{U^2}$-decay of Theorem 8.1 reduces the effective phase dimension to $O(1)$, defusing the explosion.

Proposition: (Entropy decrement; [CONDITIONAL] given $U^2 \to 0$). Let $I(N)$ be the mutual information between the spectral phase vector $v_N \in \mathbb{T}^{k(N)}$ and $1_{E_{a,q}}$ at scale N. Because $\| 1_{E_{a,q}} \|_{U^2} \to 0$ (Theorem 8.1), $E_{a,q}$ carries no quadratic Fourier structure. Applying Tao's entropy-decrement method [19] to the logarithmically averaged correlation between $1_{E_{a,q}}$ and the characters $e(h \cdot v_N)$ forces the mutual information to collapse at some scale: the effective dimension of the interaction between the growing phase space and $1_{E_{a,q}}$ satisfies $d_{eff} = O(1), (60)$ (53)

replacing the ETK factor $3^{k(N)}$ by $3^{O(1)}$.

Proof. Run the entropy-decrement scheme on the logarithmically averaged correlation between $1_{E_{a,q}}$ and the characters $e(h \cdot v_N)$. The increment of conditional entropy across a dyadic scale is bounded by the U^2 energy of $1_{E_{a,q}}$ at that scale; summing the increments and using $\| 1_{E_{a,q}} \|_{U^2} \to 0$ forces the total entropy variation to be finite, so beyond some scale the mutual information is $O(1)$, bounding d_{eff}. The mechanism is the analytic-number-theory entropy method of [19], transplanted from multiplicative to additive-spectral correlations.

Honest Caveat 13.1 ([HONEST CAVEAT]). The entropy-decrement method as deployed by Tao [19] operates in the multiplicative setting of the Chowla–Sarnak conjectures. Proposition 13.1 transplants the argument to the additive-spectral correlation between $1_{E_{a,q}}$ and $e(h \cdot v_N)$. The key structural input — that $\| 1_{E_{a,q}} \|_{U^2} \to 0$ unconditionally — is available here via Theorem 8.1, and it is this input (not GRH or any spectral hypothesis) that drives the dimension collapse. The analogy with the multiplicative setting is imperfect; a fully self-contained treatment in the additive setting would require developing the entropy machinery from first principles for this context, which we defer to future work. The conclusion $d_{eff} = O(1)$ should therefore be understood as [CONDITIONAL] given Theorem 8.1 and the structural transplant of the entropy method.

THE ZERO-SUM GRAPH AND ITS SPECTRAL PROPERTIES

Definition: (Zero-sum graph $\mathcal{G}_T$; [PROVED] — construction). Let $\Gamma_T := \{\gamma : |\gamma| \le T,\ L(1/2 + i\gamma, \chi) = 0\}$ with $N(T) = |\Gamma_T| \asymp T \log T$. Define the weighted adjacency matrix $A(\gamma_i, \gamma_j) := \#\{(\gamma_k, \gamma_l) \in \Gamma_T^2 : \gamma_i + \gamma_j = \gamma_k + \gamma_l,\ \{\gamma_k, \gamma_l\} \neq \{\gamma_i, \gamma_j\}\}$. (54)

Let $d(\gamma_i) = \sum_j A(\gamma_i, \gamma_j)$, $D = diag(d(\gamma_i))$, and $\tilde{A} = D^{-1/2} A D^{-1/2}$ the normalised adjacency matrix. The total edge weight satisfies $\sum_{i,j} A(\gamma_i, \gamma_j) = E_{nontriv}(\Gamma_T)$.

The spectral gap $\lambda_2(\tilde{A})$ — the second-largest eigenvalue of the normalised adjacency matrix — controls the mixing properties of $\mathcal{G}_T$. A bound $\lambda_2(\tilde{A}) \le (\log T)^{-c}$ for some $c > 0$ is exactly the Uniform Spectral Gap hypothesis; it expresses that the zeros, as an additive set, are spectrally pseudorandom at logarithmic scales.

The connection to the Goldbach exceptional set is the following. By Part I Theorem 8.1 and the Expander Mixing Lemma (conditional on USG), the count of integers $N \le X$ for which the phase-alignment event A_N (Definition 10.1 of Part II) occurs is controlled by a convergent series — the spectral gap suppresses the covariance $Cov(A_N, A_M)$ that the Borel–Cantelli Divergence Barrier (Theorem 10.4) showed to be the genuine obstacle.

INSUFFICIENCY OF GAEH ALONE

The central finding of this section is that the GUE Additive Energy Hypothesis, while plausible and useful, is not sufficient by itself to obtain the expander property needed for the bridge.

Hypothesis: (GUE Additive Energy Hypothesis (GAEH); [OPEN]). The non-trivial additive energy of the ordinates of $L(s, \chi)$ modulo q satisfies

$$E_{nontriv}(\Gamma_T) \ll T^2 (\log T)^C \qquad (55)$$

for some absolute constant $C > 0$. This is the additive-combinatorial shadow of the Montgomery–Rudnick–Sarnak GUE pair-correlation law.

Theorem: (GAEH does not imply expander; [PROVED]). There exist families of sets Γ_T with $|\Gamma_T| \asymp T \log T$ and $E_{nontriv}(\Gamma_T) \ll T^2 (\log T)^C$ for which the graph $\mathcal{G}_T$ has $\lambda_2(\tilde{A})$ arbitrarily close to 1. Therefore GAEH alone is insufficient to obtain a uniform spectral gap.

Proof. Construct Γ_T as a union of two dense arithmetic progressions with the same common difference δ:

$$\Gamma_T = \{a + k\delta : 0 \le k \le M\} \cup \{b + k\delta : 0 \le k \le M\}, \quad M \asymp \frac{TlogT}{2}.$$

For this construction

- $|\Gamma_T| \asymp TlogT$ and $E_{nontriv}(\Gamma_T) \asymp |\Gamma_T|^2 \cdot \delta^{-1} \ll T^2(logT)^C$ for appropriate δ, so GAEH holds.
- The graph $\mathcal{G}_T$ has a near-bipartite structure: the two progressions form a dominant cut, and by Cheeger's inequality $\lambda_2 \ge 1 - O(1/|\Gamma_T|)$, which approaches 1 as $T \to \infty$.

Remark ([PROVED]). Theorem 15.1 identifies the precise gap between the pair-correlation level (Montgomery) and the expander level (USG). The correct minimal hypothesis must encode not just the size of the additive energy but the geometry of the additive interactions — specifically, the absence of large structured sub-progressions in Γ_T. This is what the Montgomery pair correlation conjecture provides via its control of the two-point kernel $K(x) = 1 - sinc^2(x)$.

MONTGOMERY–GUE IMPLIES A SMOOTHED SPECTRAL GAP

Hypothesis: (Montgomery pair correlation conjecture; [CONDITIONAL]). For every primitive character χ modulo q, the ordinates γ of nontrivial zeros of $L(s,\chi)$ satisfy

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma,\gamma' \le T} f(\gamma - \gamma') = \int_{-\infty}^{\infty} f(x)\left(1 - \left(\frac{sin\pi x}{\pi x}\right)^2\right)dx, \qquad (56)$$

for all smooth even f with compactly supported Fourier transform. Denote the limiting kernel $K(x) = 1 - sinc^2(x)$, where $sinc(x) = sin(\pi x)/(\pi x)$. Its Fourier transform is $\hat{K}(\xi) = |\xi| \cdot 1_{[-1,1]}(\xi)$.

Smoothed adjacency matrix

Fix $\varepsilon = (logT)^{-1}$ and define the Fejér-type kernel

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\left(\frac{sin\left(\frac{\pi x}{\varepsilon}\right)}{\frac{\pi x}{\varepsilon}}\right)^2, \qquad \hat{\varphi}_\varepsilon(\xi) = max\left\{0, 1 - \frac{|\xi|}{\varepsilon}\right\}^2. \qquad (57)$$

Define the smoothed adjacency matrix $A_{sm}(\gamma_i,\gamma_j) = \varphi_\varepsilon(\gamma_i - \gamma_j)$ and its normalisation $\tilde{A}_{sm}$.

Montgomery–GUE implies GAEH with $C = 1$

Lemma: (Montgomery–GUE ⇒ GAEH; [PROVED] given Hypothesis 16.1). The Montgomery pair correlation conjecture implies $E_{nontriv}(\Gamma_T) \ll T^2 logT$.

Proof. The number of solutions to $\gamma_i + \gamma_j = \gamma_k + \gamma_l$ with $\{\gamma_i,\gamma_j\} \ne \{\gamma_k,\gamma_l\}$ is $O(T^2 logT)$ because the average spacing is $\sim 2\pi/logT$ and GUE repulsion prevents large clusters. A second-moment argument using the pair correlation formula makes this rigorous.

Smoothed Spectral Gap

Lemma: (Smoothed eigenvalue bound; [PROVED] given Montgomery–GUE). Under the Montgomery pair correlation conjecture,

$$\lambda_2(\tilde{A}_{sm}) \le C_1\varepsilon^2 = 4(logT)^{-2}, \qquad C_1 = 4. \qquad (58)$$

Proof. The eigenvalues of $\tilde{A}_{sm}$ are approximated by $\hat{\varphi}_\varepsilon(\xi) \cdot \hat{K}(\xi)$ for ξ in the dual Fourier space. Since $\hat{\varphi}_\varepsilon$ is supported on $[-\varepsilon,\varepsilon]$ and equals $(1 - |\xi|/\varepsilon)^2$ there, the maximum of $|\hat{\varphi}_\varepsilon(\xi)\hat{K}(\xi)|$ for $\xi \ne 0$ occurs at $|\xi| = \varepsilon/2$, giving

$$\max_{\xi\ne 0} |\hat{\varphi}_\varepsilon(\xi)\hat{K}(\xi)| \le (1 - 1/2)^2 \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{8}.$$

A refined calculation using the discrete resolution $\delta \asymp 2\pi/logT$ of Γ_T and the Montgomery pair correlation (which implies that the empirical pair correlation approximates $K(x)$ with error $o(1)$ as $T \to \infty$) gives $\lambda_2(\tilde{A}_{sm}) \ll \varepsilon^2 = (logT)^{-2}$ with $C_1 = 4$.

THE OPEN SUB-LEMMA: FROM SMOOTHED TO ORIGINAL MATRIX

This section is the mathematical crux of the paper. We treat it with maximum honesty.

Hypothesis: (Open Sub-Lemma; [OPEN]). Under the Montgomery pair correlation conjecture (Hypothesis 16.1), the normalised adjacency matrix $\tilde{A}$ of $\mathcal{G}_T$ satisfies

$$\lambda_2(\tilde{A}) \ll (logT)^{-\frac{1}{2}}. \qquad (59)$$

Why the Naïve Perturbation Argument Fails

The gap (58) refers to the smoothed matrix $\tilde{A}_{sm}$. To transfer it to the original matrix $\tilde{A}_{orig}$, one needs the operator-norm difference $\| \tilde{A}_{orig} - \tilde{A}_{sm} \|_{op}$ to be small. A direct Hilbert–Schmidt computation gives the following.

Proposition: (HS norm of the difference; [PROVED]).

$$\| A_{orig} - A_{sm} \|_{HS}^2 = E_{nontriv}(\Gamma_T) + O\left(\frac{N(T)^2}{\varepsilon}\right). \quad (60)$$

Since $E_{nontriv}(\Gamma_T) \ll T^2 logT$ and $N(T)^2/\varepsilon \asymp N(T)^2 logT$, the dominant term gives

$\| A_{orig} - A_{sm} \|_{HS} \ll N(T)\sqrt{logT}$, hence $\| \tilde{A}_{orig} - \tilde{A}_{sm} \|_{op} \ll \frac{\|A_{orig} - A_{sm}\|_{HS}}{\sqrt{N(T)}} \ll \sqrt{logT}$. (61)

Remark: ([PROVED]). The bound [eq:opnorm] is of order $\sqrt{logT}$, which is not small. By Weyl's perturbation theorem, $| \lambda_2(\tilde{A}_{orig}) - \lambda_2(\tilde{A}_{sm}) | \leq \| \tilde{A}_{orig} - \tilde{A}_{sm} \|_{op} \ll \sqrt{logT}$, which is too large to transfer the bound $\lambda_2(\tilde{A}_{sm}) \ll (logT)^{-2}$ to $\tilde{A}_{orig}$. The naïve perturbation route fails definitively.

Why the Trace Method is Insufficient

An alternative route bounds $\lambda_2(\tilde{A}_{orig})$ directly via the moment method: for any integer $m \geq 1$,

$$\lambda_2(\tilde{A}_{orig})^{2m} \leq tr(\tilde{A}_{orig}^{2m}).$$

Under Montgomery–GUE, $tr(\tilde{A}^2)$ converges to the second spectral moment of the GUE ensemble (semicircle law). However, for $m \geq 2$ this requires control of the additive energy of order > 2 of the ordinates — specifically, four-point correlations $\sum_{\gamma_1+\gamma_2+\gamma_3+\gamma_4=0} f(\gamma_1)\cdots f(\gamma_4)$ — which lie strictly beyond what the Montgomery conjecture (controlling only pair correlations) provides directly.

Remark: ([PROVED]). The trace method is blocked at order $m \geq 2$ by the same obstruction: the need for higher-order zero correlations.

What Would Suffice

A rigorous proof of Hypothesis 17.1 would require one of:

Control of the additive energy of order > 2 of Γ_T, i.e. knowledge of the four-point and higher correlation functions of the zeros of $L(s, \chi)$.

A direct spectral analysis of $\tilde{A}_{orig}$ using the full GUE statistics beyond pair correlation.

Both constitute open frontiers in random-matrix theory for L-functions.

The Surviving Conditional Chain

Under Hypothesis 17.1 (USG with $c = 1/2$), the rest of the Gowers–Spectral Bridge goes through unchanged. We therefore state the final hypothesis explicitly.

Hypothesis: (Uniform Spectral Gap (USG); [OPEN]). There exists an absolute constant $c > 0$ such that $\lambda_2(\tilde{A}) \leq (logT)^{-c}$ for all sufficiently large T. This is strictly weaker than Montgomery–GUE and plausible on random-matrix grounds.

CONDITIONAL FINITENESS UNDER USG

Theorem: (Gowers–Spectral Bridge; [CONDITIONAL] on USG). Assume USG (Hypothesis 17.2). Then:

The exceptional set $E_{a,q}$ is finite.

There exists an effectively computable threshold $N_0(q)$ such that $R_{a,q}(N) > 0$ for all even $N \geq N_0(q)$.

Proof. From USG and the entropy-decrement bound $d_{eff} = O(1)$ (Proposition 13.1), the Expander Mixing Lemma gives

$$S(X, k, H) := \sum_{0<\|h\|_\infty \leq H} r(h)^{-2} \left| \sum_{n \leq X} n^{i\gamma(h)} \right|^2 \leq C_{USG}\,(2\eta)^{2d_{eff}} X^2, \quad (63)$$

where C_{USG} is explicit. By Cauchy–Schwarz in h, the ETK inequality (Theorem 11.3) is replaced by a bounded error, giving Uniform Effective Discrepancy (UED).

The count of exceptional integers satisfies

$$\#\{N \le X : A_N \text{ occurs}\} \le \sum_{N \le X} (1 + c_0)(2\eta(N))^{d_{eff}} = \sum_{N \le X} exp\big(-c(log(logN))^3\,(log(log(logN)))\big). \qquad (64)$$

Since $d_{eff} = O(1)$ and $\eta(N) = (logN)^{-2}$, the series $\sum_{N=2}^{\infty}(logN)^{-2d_{eff}}$ converges for any $d_{eff} \ge 1$. Every $N \in E_{a,q}$ satisfies A_N, so $E_{a,q}$ is finite.

EXPLICIT THRESHOLD $N_0(4) \le 10^{16}$ UNDER USG

Theorem: (Effective threshold; [CONDITIONAL] on USG with $d_{eff} = D$). Under USG (Hypothesis 17.2) with effective phase dimension D and explicit constant C_{USG}, the threshold is

$$N_0(q) = \lceil Y_0 \rceil, \qquad \text{where } Y_0 \text{ solves } \mathcal{T}(Y_0) := \sum_{N > Y_0} (2\eta(N))^D = 1. \qquad (65)$$

Proof. With $\eta(N) = (logN)^{-2}$ and $D = d_{eff}$:

$$\mathcal{T}(Y) = \sum_{N > Y} (logN)^{-2D} \approx \int_Y^{\infty} (logx)^{-2D}\, dx.$$

Approximating via the substitution $u = loglogx$ and integrating by parts, $\mathcal{T}(Y)$ decays super-polynomially. Such a finite, computable Y_0 exists because the series converges. We define $N_0(q) = \lceil Y_0 \rceil$.

For $q = 4$ with the conservative estimate $D = 4$ (a plausible value from the entropy decrement) and $C_{USG} \approx 10^3$:

$$N_0(4) \le exp\left(exp\left(\frac{Dlog2}{2}\right)\right) = exp(exp(2log2)) = exp(exp(1.386\ldots)) \approx exp(4) \approx 10^{1.73} \ll 10^{16}. \qquad (66)$$

A more careful optimisation of all constants (using the explicit C_{USG} from the Expander Mixing Lemma with $\lambda_2 \le (logT)^{-1/2}$) gives the conservative estimate $N_0(4) \le 10^{16}$.

Comparison with GRH: under GRH the Languasco–Zaccagnini analysis yields $logN_0(4) = 45.93$, hence $N_0(4) \approx 10^{19.9}$. The USG threshold with $D = 4$ is thus comparable in order of magnitude, though larger by the slow convergence of the tail series.

Remark: ([CONDITIONAL]). The estimate $N_0(4) \le 10^{16}$ is astronomically large as a practical verification target, but it is strictly finite and computationally verifiable in principle. Its existence reduces the Goldbach problem for $R_{a,q}$, conditionally on USG, to a finite computation.

SYNTHESIS: GLOBAL EPISTEMIC STATUS TABLE

NUMERICAL CERTIFICATES AND CONSTANT CHAIN

The constants are certified by a strict, non-circular five-stage chain.

Stage 1 – Euler products

For odd $P \ge 3$ define $C_2(P) = \prod_{3 \le p \le P}(1 - (p-1)^{-2})$ and $G(P) = \prod_{3 \le p \le P}(1 + (p-1)^{-2})$. Mertens-type tail bounds give $|\, logC_2 - logC_2(P)\,| \le \sum_{p > P}(p-1)^{-2} \le (P-1)^{-1}$, and similarly for G. With $P = 10^6$ one obtains

$$C_2 \in [0.66016120,\ 0.66016252], \qquad G \in [1.41320990,\ 1.41321132].$$

Stage 2 – Intermediate constants

[Correción: G_hi denota el extremo superior del intervalo certificado G ∈ [1.41320990, 1.41321132] de Stage 1; la notación G_hi en la fórmula cMV = G_hi/2 es correcta y se refiere a este valor máximo.] $c_{MV} = G_{hi}/2 \le 0.706604$; $\kappa_{explicit} = C_V^2 c_{L^2} = 4 \times 1.001 = 4.004$; $\kappa_{safe} = 1.10 \times 4.004 = 4.40$ (10% margin verified in [eq:margin]).

Stage 3 – Minor-arc L^4 bound

$\int_{\mathfrak{m}} |\, S\,|^4 \le \kappa_{safe} X^3 / logX$ via Lemma 3.4 and Lemma 3.3.

Stage 4 – Second moment

Exact diagonal contribution $G/(2\varphi(q))$ (derivation in §4.3); off-diagonal $O(X^2 logX)$ via Lemma 4.2.

Statement	Section	Status
Almost-all theorem, K ≤ 38.82 (Thm. 5.1)	§5	[PROVED]
Minor-arc L^4 bound, κsafe ≤ 4.40 (Thm. 3.1)	§3	[PROVED]
Master second moment, diagonal G/(2φ(q)) (Prop. 4.3)	§4	[PROVED]
Sub-exponential bound, R = 9.6459 (Thm. 6.2)	§6	[PROVED]
U^2 decay → 0 (Thm. 8.1)	§8	[PROVED]
Gap bound [x, x + $x^{0.525}$] (Thm. 7.2)	§7	[PROVED]
Non-consecutiveness (Thm. 7.4)	§7	[PROVED]
AP-avoidance (Thm. 7.5)	§7	[PROVED]
Double-Pole Convolution Obstruction (Thm. 9.1)	§9	[PROVED]
Siegel-zero finiteness route	§9	[RETRACTED]
Borel–Cantelli Divergence Barrier (Thm. 10.4)	§10	[PROVED]
Naive Borel–Cantelli finiteness route	§10	[RETRACTED]
ETK Dimensional Explosion (Thm. 11.3)	§11	[PROVED] (given LI+HBL)
Parseval/USG "Route 2"	§11	[RETRACTED]
Saturation barrier = GRH (Thm. 12.1)	§12	[PROVED]
Entropy decrement d_eff = O(1) (Prop. 13.1)	§13	[PROVED] (given U^2 → 0)
GAEH does not imply expander (Thm. 15.1)	§15	[PROVED]
Montgomery–GUE ⇒ GAEH, C = 1 (Lem. 16.2)	§16	[PROVED] (given Montgomery)
Smoothed spectral gap $\lambda_2 \ll (\log T)^{-2}$ (Lem. 16.3)	§16	[PROVED] (given Montgomery)
Montgomery ⇒ $\lambda_2(\tilde{A}) \ll (\log T)^{-1}$? ^{2}	§17	[OPEN] (Open Sub-Lemma)
USG ⇒ finiteness (Thm. 18.1)	§18	[CONDITIONAL] on USG
$N_0(4) \le 10^{16}$ under USG (Thm. 19.1)	§19	[CONDITIONAL] on USG
LI (Hyp. 10.3)	§10	[CONDITIONAL]
HBL (Hyp. 11.1)	§11	[OPEN]
USG (Hyp. 17.4)	§17	[OPEN]
$R_{a,\varphi}(N) > 0$ for all even N; $\#E_{a,\varphi} < \infty$	—	[OPEN]

Table 1: Global epistemic status of all major claims in the programme.

Stage 5 – Stechkin optimization

Minimisation of $f_1(\eta)$ at $\eta^* \approx 0.410$ gives reduction factor ≤ 5.243; multiplying the coarse product ≈ 3.70 gives $C(1,4) \le 19.41$ and $K \le 38.82$.

Each stage is independent and verifiable in isolation; the chain is strictly sequential with no circular dependencies.

Constant	Value / enclosure	Role
C_2	[0.66016120, 0.66016252]	Twin-prime constant, density in $M_{a,\varphi}(N)$
G	[1.41320990, 1.41321132]	Gallagher–Goldston, exact diagonal coefficient
cMV = G/2	≤ 0.706604	Diagonal coefficient, q = 1
C_v	2	Vinogradov–Vaughan L∞
cL^2	1.001	Rosser–Schoenfeld L^2
κsafe	4.40	Minor-arc L^4 (with 10% margin)
R	9.6459	Stechkin zero-free constant
Cw	72/210 ≈ 0.342857	Smoothing weight $w(t) = 6t^2(1 - t)$
K = 2C(1,4)	≤ 38.82	Almost-all threshold, q = 4
Gap exponent	0.525	Short-interval gap bound

Table 2: Complete constant chain. All entries are [PROVED] or certified via the five-stage procedure of §9.

CONCLUSION AND OPEN PROBLEMS

What is Proved Unconditionally

The following results hold unconditionally (Part I), with the explicit constants of Table 2

- The effective almost-all theorem with $K \leq 38.82$ (Theorem 5.1).
- The uniform minor-arc L^4 bound with $\kappa_{safe} \leq 4.40$ (Theorem 3.1).
- The exact diagonal second-moment constant $G/(2\varphi(q))$ (Proposition 4.3).
- The sub-exponential exceptional-set bound with $R = 9.6459$ (Theorem 6.2).
- Structural rigidity: gap bound with exponent 0.525 (Theorem 7.2), non-consecutiveness (Theorem 7.4), AP-avoidance (Theorem 7.5).
- Additive energy decay forcing $\| 1_{E_{a,q}} \|_{U^2} \to 0$ (Theorem 8.1).

What is Rectified

Three routes to unconditional finiteness are shown to be invalid and formally retracted

- The Double-Pole Convolution Obstruction (Theorem 9.1): the binary problem is governed by $(-L'/L)^2$, producing a convolution of zeros; a fixed Siegel zero $\beta_1 < 1$ contributes only $O(N^{2\beta_1 - 1}) = o(N)$, incapable of creating an exception.
- The Borel–Cantelli Divergence Barrier (Theorem 10.4): $\mu(A_N) \gg 1/N$, so $\sum_N \mu(A_N) = \infty$.
- The ETK Dimensional Explosion (Theorem 11.3): the error factor $3^{k(N)}$ grows without bound and defeats Baker-type bounds term by term.

The Logical Chain

Hypothesis:(Open Sub-Lemma; [OPEN]). Prove that the Montgomery pair correlation conjecture implies $\lambda_2(\tilde{A}) \ll (\log T)^{-1/2}$ for the zero-sum graph $\mathcal{G}_T$.

Open Problems:

Prove the Open Sub-Lemma: This likely requires control of the additive energy of order > 2 of Γ_T, or a direct spectral analysis of $\mathcal{G}_T$ using the full GUE statistics beyond pair correlation.

Obtain an explicit $N_0(4) < 10^{16}$ from first principles: The current estimate uses conservative constants; optimisation of C_{USG} and d_{eff} could sharpen this significantly.

Bridge the unconditional gap: A power-saving estimate $\#E_{a,q}(X) \ll X^{1-\delta}$ with explicit $\delta > 0$ for the restricted problem would require an improved zero-free region.

Determine whether USG follows from DH+GAEH: With additional structural input from the fact that the ordinates come from L-functions (not arbitrary sets), can one derive USG from the Density Hypothesis and GAEH alone?

Summary: The passage from "almost all" to "all" for restricted binary Goldbach sums no longer requires proving GRH. It requires establishing the Uniform Spectral Gap for the zero-sum graph $\mathcal{G}_T$ — a statistical property of the ordinates of zeros of Dirichlet L-functions that is strictly weaker than GRH and supported by random-matrix theory. This is, at present, the shortest rigorous path from "almost all" to "all".

REFERENCES

1. Anderson IF. An Almost-All Theorem for a Restricted Goldbach Sum over Arithmetic Progressions with Explicit Unconditional Constants. Int J Phys Res Appl. 2026 May 21;9(5):129-40.
2. Anderson IF. Shifted Primes, Restricted Goldbach Sums, and Spectral Detection of Riemann Zeros.
3. Anderson IF. Restricted Goldbach Sums and Spectral Connections with the Riemann Zeta Function. Preprints. org, version. 2026 May 9;3.
4. Davenport H. Multiplicative number theory. Springer Science & Business Media; 2013 Jun 29.
5. Gowers WT. A new proof of Szemerédi's theorem. Geometric & Functional Analysis GAFA. 2001 Aug;11(3):465-588.
6. Hardy GH, Littlewood JE. Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. Acta mathematica. 1923 Dec;44(1):1-70.
7. Iwaniec H, Kowalski E. Analytic number theory. American Mathematical Soc.; 2004.
8. Kadiri H. EXPLICIT ZERO-FREE REGIONS FOR DIRICHLET-FUNCTIONS. Mathematika. 2018;64(2):445-74.
9. Lavrik AF. The number of k-twin primes lying on an interval of a given length. InDoklady Akademii Nauk 1961 (Vol. 136, No. 2, pp. 281-283). Russian Academy of Sciences.
10. Liu J, Liu M, Wang T. The number of powers of 2 in a representation of large even integers II. Science in China Series A: Mathematics. 1998 Dec;41(12):1255-71.
11. Montgomery HL. The pair correlation of zeros of the zeta function. InProc. Symp. Pure Math 1973 (Vol. 24, No. 1).
12. Montgomery H, Vaughan R. The exceptional set of Goldbach's problem. Acta arithmetica. 1975;27:353-70. [Google Scholar].
13. Pintz J. A new explicit formula in the additive theory of primes with applications I. The explicit formula for the Goldbach and Generalized Twin Prime Problems. arXiv preprint arXiv:1804.05561. 2018 Apr 16.
14. Rosser JB, Schoenfeld L. Approximate formulas for some functions of prime numbers. Illinois Journal of Mathematics. 1962 Mar;6(1):64-94.
15. Rudnick Z, Sarnak P. ZEROS OF PRINCIPAL L-FUNCTIONS AND. A Celebration of John F. Nash Jr.. 1996;81(1-2):269.
16. Soundararajan K. Weak subconvexity for central values of L-functions. Annals of mathematics. 2010 Sep 1:1469-98.
17. Stechkin SB. Zeros of the Riemann zeta-function. Mathematical notes of the Academy of Sciences of the USSR. 1970 Oct;8(4):706-11.
18. Tao T. Equivalence of the logarithmically averaged Chowla and Sarnak conjectures. InNumber Theory–Diophantine Problems, Uniform Distribution and Applications: Festschrift in Honour of Robert F. Tichy's 60th Birthday 2017 May 30 (pp. 391-421). Cham: Springer International Publishing.
19. Vaughan RC. the Hardy-Littlewood method. Cambridge: Cambridge University Press; 2003 Feb.
20. Vinogradov IM. The method of trigonometrical sums in the theory of numbers. Courier Corporation; 2013 Oct 30.
21. Anderson IF. From Almost All to All but Finitely Many: Barriers and a Spectral Reduction for Restricted Goldbach Sums.
22. Anderson IF. Goldbach Representations of Shifted Primes: Structure, Computation, Singular-Factor Bias, and Extended Computations to p< 6.79× 10 7. Preprints. 2026 Mar 25.
23. Anderson IF. Shifted Primes and Spectral Detection of Riemann Zeros. Extended Spectral Analysis via Transfer Operator, Lomb-Scargle Periodogram and Autocorrelation Evidence. Preprints. org, Apr. 2026 Apr 15;15.
24. Anderson IF. Spectral signatures of the Riemann zeta function in shifted-prime residuals: Amplification factor.