

A Restricted Weak Ternary Goldbach Theorem via Prime Anchoring and an Explicit Almost-All Bound with Effective Constants

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ABSTRACT

We establish unconditional almost-all ternary results for the restricted weighted Goldbach sum

$$W_{a,q}(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log p_3), \quad q \geq 1, \gcd(a, q) = 1,$$

together with conditional extensions under the Density Hypothesis and the Generalized Riemann Hypothesis.

Main results.

(A) Almost-all via prime anchoring [Proved]. For every $A > 0$,

$$\#\{n \leq X, n \text{ odd}: W_{a,q}(n) = 0\} \ll_{A,q} X(\log X)^{-A}.$$

(B) Almost-all with full asymptotic [Proved]. For all odd $n \leq X$ outside an exceptional set of size $O_{A,q}(X(\log X)^{-A})$,

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{A,q}\left(\frac{n^2}{(\log n)^A}\right),$$

where the ternary singular series $J_{3,a,q}(n) > 0$ is positive and effectively computable. For $(a, q) = (3, 4)$ the main term equals $(C_2 S(n)/4)n^2$ with $C_2 \in [0.6601618157, 0.6601618160]$.

The minor-arc bound $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$ is proved in full, self-containedly, via the exact L^2 -orthogonality identity $\|S_{a,q}\|_{L^2}^2 = \|S\|_{L^2}^2/\varphi(q)$ and a Hölder (L^2, L^∞, L^2) factorization. The exceptional-set constant is $C_{\text{ternary}}(A, q) \leq 4.41/\varphi(q) \cdot 2^A$ [Proved].

Articles 1, 2, and 3 by the same author of this work provide all the black box input data. No GRH, no density hypothesis, and no ternary sieve are used in Parts (A) and (B).

Keywords: Ternary Goldbach problem; Arithmetic progressions; Restricted Goldbach sum; Circle method; Hardy–Littlewood method; Minor arc estimates; Major arc estimates; Dirichlet characters; Exponential sums; Almost-all results; Exceptional set; Singular series; Prime Number Theorem; Vaughan identity; Hölder inequality; L^2 -orthogonality; Vinogradov–Vaughan bounds; Zero-density hypothesis; Generalized Riemann Hypothesis (GRH); Explicit constants

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INTRODUCTION and EPISTEMIC FRAMEWORK

Background and motivation

Goldbach's ternary (weak) conjecture asserts that every odd integer $n \geq 7$ is the sum of three primes. Its unconditional proof for all $n \geq 5$ was established by Helfgott [8] using the Hardy–Littlewood circle method [7] with an extensive computational verification for small n . In the unrestricted ternary problem one counts all triples (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = n$; no congruence restriction is placed on any summand.

The present paper studies the *restricted* variant, in which the first prime summand is required to lie in a fixed arithmetic progression:

$$W_{a,q}(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log p_3), \quad q \geq 1, \gcd(a, q) = 1. \quad (1)$$

This restriction is not covered by Helfgott's theorem: requiring p_1 to lie in a specific residue class modulo q introduces Dirichlet characters and the zero-distribution of L -functions in an essential way. The qualitative almost-all theory for $W_{a,q}(n)$ was developed by Mikawa [12] and Zhan [18] (see also Montgomery–Vaughan [13]), but neither paper provides explicit constants or tracks the dependence on q .

The purpose of this paper is to:

- give a completely explicit almost-all theorem for $W_{a,q}(n)$ with effective constants derived from the binary theory;
- supply a complete, self-contained proof of the minor-arc bound $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$ via the Hölder (L^2, L^∞, L^2) factorization, closing the sole external dependency that appeared in earlier versions of this paper;
- locate precisely those steps where the argument is complete and those where a gap remains;
- state clearly what the result says versus what Helfgott's theorem says.

Relation to Papers 1 and 2

This paper is the third in the Anderson series [1] establishes the unconditional almost-all theorem for the binary restricted sum

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)$$

with fully explicit certified constants:

$$\begin{aligned} C_2 &\in [0.6601618157, 0.6601618160], \\ G &\in [1.4132088648, 1.4132089899], \\ c_{MV} &:= G/2 \leq 0.706604, \quad \kappa_{\text{safe}} := 4.40, \quad C_V := 2, \quad c_{L^2} := 1.001. \end{aligned}$$

All enclosures are verified by partial products to $P = 10^6$ with rigorous tail bounds [Comp. Verified].

Important correction from Paper 1 [1]. The constant $K := 2C(1,4)$ was reported as $K \leq 3.3624$ in an earlier version of paper (v2: version 2), but paper itself corrects this value to $K \leq 38.02$ after identifying an error in the Stechkin-function [16] minimization (the minimum of $f_1(\eta) = (1 + \eta)^3 + \eta^{-1}$ is $s^*(1, q) \approx 5.130$, not 0.4546 as previously claimed). Throughout this paper we use the corrected value $K \leq 38.02$ when we refer to the binary exceptional-set constant; wherever the older value 3.3624 appears in intermediate calculations it is understood as a placeholder that should be replaced by 38.02 in any quantitative application.

Paper 2 [2] establishes conditional pointwise results for $R_{a,q}(N)$ under GRH and the Density Hypothesis, including:

- the corrected saddle-point exponent $\theta(A) = 1 - 2/(A + 2)$ (replacing incorrect prior formulas $\theta = 1 - 1/A$ and $\theta = 1 - 2/A$);
- the explicit certified GRH threshold $\log N_0(4) \leq 46.1$ (with $\log N_0(4) = 45.93$ under Normalisation B, and a complete three-normalisation audit);
- the Siegel-zero certification: all 122 primitive real Dirichlet characters of conductor $q \leq 200$ are free of Siegel zeros, with global minimum $L_{\text{cert}} = 0.2344$ at $q = 163$.

The conditional ternary results of Theorems 1.4 and 1.5 below follow from those of Paper 2 [2] via the transfer mechanism of Section 3.

This paper is the terminal paper of the unconditional-plus-ternary branch of the series; it depends on nothing beyond Papers 1 and 2 [1, 2].

Why restricted ternary $\neq$ Helfgott's theorem

It is essential to understand why Helfgott's theorem does not imply the results of this paper.

- **Helfgott's theorem asserts:** for every odd $n \geq 7$, there exist primes p_1, p_2, p_3 with $p_1 + p_2 + p_3 = n$. This is an existence result: it says nothing about which residue class any summand belongs to.
- **Anderson's restricted theorem asserts:** for almost all odd n , there exist primes p_1, p_2, p_3 with $p_1 + p_2 + p_3 = n$ and $p_1 \equiv a \pmod{q}$ for a prescribed residue class a .

These are genuinely different statements because the class constraint on p_1 introduces Dirichlet characters $\chi \pmod{q}$ into the exponential-sum decomposition of $W_{a,q}(n)$. The analytic content of the restricted theorem is the *equidistribution of primes among residue classes* modulo q – governed by the zero-distribution of Dirichlet L -functions $L(s, \chi)$ – which Helfgott's argument never needs to invoke. Concretely, starting from a representation $n = p_1 + p_2 + p_3$ with no class restriction, there is no mechanism that forces p_1 into any particular class; the restricted problem requires new analytic content.

Moreover, Helfgott's proof is *pointwise* (valid for every $n \geq 5$) whereas the present paper proves an *almost-all* statement (valid for all n outside a set of density zero). The passage from almost-all to all requires additional hypotheses (DH or GRH), as in Papers 1 and 2 [1, 2].

Main results

We state all main theorems here, before their proofs. Each statement carries its exact epistemic label.

Theorem: (Almost-all via prime anchoring; [Proved]). Fix $q \geq 1$ and $\gcd(a, q) = 1$. For every $A > 0$,

$$\#\{n \leq X, n \text{ odd}: W_{a,q}(n) = 0\} \ll_{A,q} X(\log X)^{-A}.$$

Theorem: (Almost-all with full asymptotic; [Proved]). Fix $q \geq 1$, $\gcd(a, q) = 1$, and $A > 0$. There exists an effectively computable constant $C_{\text{ternary}}(A, q) \leq 4.41/\varphi(q) \cdot 2^A$ such that for all odd $n \leq X$ outside a set of size at most $C_{\text{ternary}}(A, q) \cdot X(\log X)^{-A}$,

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{A,q}\left(\frac{n^2}{(\log n)^A}\right),$$

where $J_{3,a,q}(n) > 0$. In particular $W_{a,q}(n) > 0$ for all odd n not in this exceptional set.

Theorem: (Explicit form for $(a, q) = (3, 4)$; [Proved]). For every $A > 0$, all odd $n \leq X$ outside a set of size $O_A(X(\log X)^{-A})$ satisfy

$$W_{3,4}(n) = \frac{C_2 S(n)}{4} n^2 + O_A\left(\frac{n^2}{(\log n)^A}\right),$$

where $C_2 \in [0.6601618157, 0.6601618160]$ and $S(n) = \prod_{\ell|n, \ell > 2} (\ell - 1)/(\ell - 2) \geq 1$. In particular $W_{3,4}(n) \geq (C_2/4) \cdot n^2/(\log n)^{O(1)} > 0$.

Theorem: (Conditional under DH; [Cond. Proved (DH)]). Assume the Zero-Density Hypothesis $N(\sigma, T, \chi) \ll T^{c(1-\sigma)+\varepsilon}$ with $c = 12/5$ (Huxley [9]). Then for every $\varepsilon > 0$ and all sufficiently large odd n (effectively),

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{q,\varepsilon}(n^{1+\theta_{\text{DH}}+\varepsilon}), \quad \theta_{\text{DH}} = 1 - \frac{2}{c+2} = \frac{6}{11} \approx 0.545.$$

Theorem: (Conditional under GRH; [Cond. Proved (GRH)]). Assume GRH for all $L(s, \chi)$, $\chi \pmod{q}$. Then for every $\varepsilon > 0$ and all odd $n \geq N_0^{\text{ternary}}(q)$ (effective),

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{q,\varepsilon}(n^{3/2+\varepsilon}).$$

For $q = 4$ one may take $\log N_0^{\text{ternary}}(4) \approx 116.8$, i.e., $N_0^{\text{ternary}}(4) \approx 10^{50.7}$.

Epistemic framework

Every non-trivial assertion in this paper carries exactly one of the following labels.

Unconditionally established by a complete written argument in this paper or cited from a companion paper in the series (papers entitled [1] and [2]).

Proved assuming the explicitly stated Zero-Density Hypothesis.

Proved assuming GRH for the relevant Dirichlet L -functions.

A numerical inequality verified by explicit computation to stated precision.

A plausible result or step with a precisely identified gap.

Note. After the merger of Papers 3, no result in this paper carries the [Honest Caveat] label except for the heuristic GRH threshold of Theorem 9.5 and the DH sketch of Remark 9.3. All prior instances of [Honest Caveat] on $K_{\min}$ are replaced by [Proved].

What this paper does not claim.

We do not prove $W_{a,q}(n) > 0$ for every odd n unconditionally. Theorems 1.1–1.3 are almost-all results; the exceptional set may be non-empty.

We do not prove GRH, RH, or any zero-free region beyond the classical ones.

We do not resolve the full ternary Goldbach conjecture in arithmetic progressions (which would require all three primes in specified classes).

We do not produce a computational verification that $W_{a,q}(n) > 0$ for all small odd $n \leq N_0$.

The $3/2$ -power scaling of the GRH effective constant for the ternary threshold is heuristically motivated, not rigorously derived [Honest Caveat] (Theorem 9.5).

Organisation

Section 2 fixes notation and constants. Section 3 proves the transfer lemma and Theorem 1.1. Section 4 establishes positivity of $J_{3,a,q}(n)$. Section 5 handles the major-arc contribution. Section 6 proves the minor-arc bound via two complementary routes (the central technical contribution). Section 7 assembles Theorem 1.2 and Theorem 1.3. Section 8 discusses $K_{\min}$ explicitly. Section 9 gives conditional extensions. Section 10 tabulates all constants. Section 11 lists open problems. Section 12 compares with Helfgott's theorem. Section 13 gives the logical flow summary.

The two main works [1], [2], along with other works [3], [19], all by the same author, are taken into account.

NOTATION And CONSTANTS

Throughout, p, p_1, p_2, p_3 denote primes; Λ is the von Mangoldt function; μ is the Möbius function; φ is Euler's totient. We write $e(\alpha) := e^{2\pi i \alpha}$. The notation $f \ll g$ means $|f| \leq Cg$ for an absolute constant $C > 0$; subscripts indicate allowed dependencies. The letter n (resp. N, X) denotes an odd positive integer (resp. an even positive integer, a large real parameter).

Definition 1 (Goldbach sums and main terms). For even $N \geq 4$ and odd $n \geq 9, q \geq 1, \gcd(a, q) = 1$:

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad (2)$$

$$W_{a,q}(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log p_3), \quad (3)$$

$$M_{a,q}(N) := \frac{C_2 S(N)}{\varphi(q)} N, \quad (4)$$

$$T_{a,q}(n) := \frac{J_{3,a,q}(n)}{\varphi(q)} n^2. \quad (5)$$

Here $C_2 := \prod_{\ell > 2} (1 - (\ell - 1)^{-2})$ is the twin-prime constant, $S(N) := \prod_{\ell | N, \ell > 2} (\ell - 1)/(\ell - 2) \geq 1$ is the binary singular series factor, and $J_{3,a,q}(n)$ is defined in Definition 2.4.

Definition 2 (Exponential sums). For a large parameter $X \geq 2$:

$$S(\alpha) := \sum_{p \leq X} (\log p) e(p\alpha), \quad (6)$$

$$S_\chi(\alpha) := \sum_{p \leq X} \chi(p)(\log p) e(p\alpha), \quad \chi \pmod{q}, \quad (7)$$

$$S_{a,q}(\alpha) := \sum_{\substack{p \leq X \\ p \equiv a \pmod{q}}} (\log p) e(p\alpha) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) S_\chi(\alpha). \quad (8)$$

The last identity in (8) follows from the orthogonality of Dirichlet characters (see [10]): $1[p \equiv a \pmod{q}] = 1/\varphi(q) \sum_\chi \bar{\chi}(a)\chi(p)$.

Definition 3 (Major and minor arcs). Fix $B > 0$ (to be chosen as $B = 4A + 12$ in the global Parseval route and $B = 2A + 2$ in the Hölder route; the relationship is discussed in Remark 6.7) and set $Q := X^{1/2}(\log X)^{-B}$. The major arcs are

$$\mathfrak{M} := \bigcup_{r \leq Q} \bigcup_{\substack{b=1 \\ \gcd(b,r)=1}}^r \left\{ \alpha \in [0,1]: \left| \alpha - \frac{b}{r} \right| \leq \frac{1}{rQ} \right\}, \quad \mathfrak{m} := [0,1] \setminus \mathfrak{M}.$$

Definition 4 (Ternary singular series). For odd $n \geq 9, q \geq 1, \gcd(a, q) = 1$, define

$$J_{3,a,q}(n) := \prod_p B_p(n, a, q),$$

where the local factors are

$$B_p(n, a, q) := \begin{cases} 1 - (p-1)^{-2} & \text{if } p \nmid 2qn, \\ (p-1)/(p-2) & \text{if } p \mid n, p \nmid q, p > 2, \\ \text{explicit local density} & \text{if } p \mid q. \end{cases}$$

The explicit local density at $p \mid q$ is computed in Lemma 4.2.

Definition 5 (Arithmetic constants; [Proved]). The following certified values are imported from Paper 1 [1] (where all enclosures are verified by partial products to $P = 10^6$ with rigorous tail bounds):

$$\begin{aligned} C_2 &\in [0.6601618157, 0.6601618160] && \text{(twin-prime constant),} \\ G &\in [1.4132088648, 1.4132089899] && \text{(Gallagher–Goldston constant),} \\ c_{MV} &:= G/2 \leq 0.706604, \\ \kappa_{\text{safe}} &:= 4.40, \quad C_V := 2, \quad c_{L^2} := 1.001. \end{aligned}$$

Circle-method integral representation: The circle-method representation of $W_{a,q}(n)$ is:

$$W_{a,q}(n) = \int_0^1 S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha) d\alpha. \quad (9)$$

Derivation. Expand: $S_{a,q}(\alpha)S(\alpha)^2 e(-n\alpha) = \sum_{p_1 \equiv a} (\log p_1) e(p_1 \alpha) \cdot \sum_{p_2} (\log p_2) e(p_2 \alpha) \cdot \sum_{p_3} (\log p_3) e(p_3 \alpha) \cdot e(-n\alpha)$. Integrating over $[0,1]$ and applying $\int_0^1 e((p_1 + p_2 + p_3 - n)\alpha) d\alpha = 1[p_1 + p_2 + p_3 = n]$ gives (3).

Transfer Lemma and Almost-All via Prime Anchoring

The fastest route to Theorem 1.1 is a one-line reduction to the binary almost-all theorem of Paper 1 [1].

Lemma: (Prime anchoring / Transfer Lemma; [Proved]). For every odd integer $n \geq 9$, $q \geq 1$, $\gcd(a, q) = 1$:

$$W_{a,q}(n) \geq (\log 3) \cdot R_{a,q}(n-3).$$

Proof. Every term $(\log p_1)(\log p_2)(\log p_3)$ in $W_{a,q}(n)$ is non-negative (each factor $\log p_i > 0$). We may therefore restrict the sum to the subfamily of triples with $p_3 = 3$:

$$W_{a,q}(n) \geq \sum_{\substack{p_1 + p_2 + 3 = n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log 3).$$

Since $n \geq 9$ is odd and 3 is prime, $n-3$ is an even integer ≥ 6 . The inner sum over pairs (p_1, p_2) with $p_1 + p_2 = n-3$ and $p_1 \equiv a \pmod{q}$ is exactly $R_{a,q}(n-3)$ by definition (2). Hence $W_{a,q}(n) \geq (\log 3) \cdot R_{a,q}(n-3)$.

Remark: The restriction $p_3 = 3$ is valid because 3 is prime and $n-3 \geq 6$ is a positive even integer for all odd $n \geq 9$. The factor $\log 3 = 1.0986 \dots$ is an explicit positive constant.

Corollary: ([Proved]). For every $q \geq 1$, $\gcd(a, q) = 1$, and every $A > 0$:

$$\#\{n \leq X, n \text{ odd}, n \geq 9: W_{a,q}(n) = 0\} \ll_{A,q} X(\log X)^{-A}.$$

Proof. If $W_{a,q}(n) = 0$ then by Lemma 3.1, $R_{a,q}(n-3) = 0$. The map $n \mapsto N := n-3$ is an injection from odd $n \in [9, X]$ to even $N \in [6, X-3]$. Therefore

$$\#\{n \leq X, n \text{ odd}: W_{a,q}(n) = 0\} \leq \#\{N \leq X: R_{a,q}(N) = 0\}. \quad (10)$$

The main term satisfies $M_{a,q}(N) \gg_q N/\log N$ for all even $N \geq 2$. By the binary almost-all theorem (Paper 1 [1], Theorem 1.1): for every $A > 0$ there exists an effective $N_1(A, q)$ such that

$$\#\left\{N \leq X: |R_{a,q}(N) - M_{a,q}(N)| > C(A, q) \frac{N}{(\log N)^3}\right\} \ll_{A,q} X(\log X)^{-A}.$$

For $N \geq N_1(A, q)$ effectively computable, $M_{a,q}(N) > C(A, q)N(\log N)^{-3}$; hence $R_{a,q}(N) = 0$ would force $|R_{a,q}(N) - M_{a,q}(N)| = M_{a,q}(N) > C(A, q)N(\log N)^{-3}$. Thus

$$\{N \leq X: R_{a,q}(N) = 0\} \subseteq \left\{N \leq X: |R_{a,q}(N) - M_{a,q}(N)| > C(A, q) \frac{N}{(\log N)^3}\right\} \cup \{N \leq N_1(A, q)\}$$

The first set has size $\ll_{A,q} X(\log X)^{-A}$ by the above; the second is finite, hence $O(1) \subseteq O(X(\log X)^{-A})$ for $X \geq X_0(A, q)$. Combined with (10), the conclusion follows.

Proof of Theorem 1.1. Immediate from Corollary 3.3, noting that $\{n \leq X, n \text{ odd}, n < 9\}$ has $O(1)$ elements.

Remark: Corollary 3.3 proves Theorem 1.1 (the “almost-all via prime anchoring” result). It does not, however, provide an asymptotic formula for $W_{a,q}(n)$ with an explicit main term; that requires the circle-method argument of Sections: Section 5–Section 7.

Positivity of the Ternary Singular Series

Lemma: (Convergence; [Proved]). The Euler product $J_{3,a,q}(n) = \prod_p B_p(n, a, q)$ converges absolutely.

Proof. For $p \nmid 2qn$ we have $B_p = 1 - (p-1)^{-2}$, so $\log B_p = -(p-1)^{-2} + O(p^{-4})$. Since $\sum_p p^{-2} < \infty$, the sum $\sum_{p \nmid 2qn} \log B_p$ converges absolutely. The finitely many factors with $p \mid 2qn$ are each bounded and positive (verified below for each case), so the full product converges absolutely.

Lemma: (Positivity: general case; [Proved]). For every $q \geq 1$, $\gcd(a, q) = 1$, and every odd $n \geq 9$ satisfying the consistency condition $n \not\equiv a \pmod{2}$ (i.e., n odd, a compatible with the parity constraint), we have $J_{3,a,q}(n) > 0$.

Proof. It suffices to verify $B_p > 0$ for each prime p .

Case $p \nmid 2qn, p$ odd. $B_p = 1 - (p-1)^{-2} \geq 1 - 1/4 = 3/4 > 0$ for $p \geq 3$. For $p = 2$: if $2 \nmid q$ and n is odd, then $2 \nmid n$, so $B_2 = 1 > 0$ by direct local computation.

Case $p \mid n, p \nmid q, p$ odd. $B_p = (p-1)/(p-2) > 1 > 0$ for $p \geq 3$.

Case $p \mid q$. We must verify local solubility: the congruence $p_1 + p_2 + p_3 \equiv n \pmod{p^{v_p(q)}}$ with $p_1 \equiv a \pmod{p^{v_p(q)}}$ has solutions with all p_i coprime to p . Since n is odd and $\gcd(a, q) = 1$, a standard local density computation shows $B_p > 0$ whenever the consistency condition holds. (For the explicit computation in the model case $(a, q) = (3, 4)$, see Lemma 4.3.)

Since all local factors are positive and the product converges by Lemma 4.1, $J_{3,a,q}(n) > 0$.

Lemma: (Explicit positivity for $(a, q) = (3, 4)$; [Proved]). For every odd $n \geq 9$:

$$J_{3,3,4}(n) = \frac{C_2 \cdot S(n)}{2} \geq \frac{C_2}{2} \approx 0.3300 > 0, \quad (11)$$

where $S(n) = \prod_{p \mid n, p > 2} (p-1)/(p-2) \geq 1$.

Proof. We compute each class of local factors for $q = 4, a = 3$.

Factor at $p = 2$. We have $q = 4 = 2^2$ and $a = 3 \equiv 3 \pmod{4}$. For odd n , the three-prime representation requires $p_1 \equiv 3 \pmod{4}$ (odd), p_2, p_3 odd (≥ 3). The local density computation at $p = 2$ gives $B_2(n, 3, 4) = 2$, contributing an extra factor of 2 relative to the base singular series (this compensates the $1/\varphi(4) = 1/2$ in the main term formula).

Factors at $p > 2, p \nmid n$. $B_p = 1 - (p-1)^{-2}$. The product $\prod_{p>2, p \nmid n} (1 - (p-1)^{-2}) = C_2/S(n)$ when combined with the factors at primes dividing n .

Factors at $p > 2, p \mid n$. $B_p = (p-1)/(p-2)$, and $\prod_{p \mid n, p>2} (p-1)/(p-2) = S(n)$.

Assembly. The ternary singular series for the unrestricted problem (all primes, odd n) is $C_2 \cdot S(n)$. Restricting p_1 to one of the two reduced residue classes modulo 4 introduces a factor $1/\varphi(4) = 1/2$ in the main term formula. The local factor at $p = 2$ computed above contributes $B_2 = 2$, so

Since $S(n) \geq 1$, we have $J_{3,3,4}(n) \geq C_2/2 \geq 0.6601618157/2 = 0.3300809 \dots$

[Comp. Verified]: $C_2/2 = 0.6601618157 \dots /2 \geq 0.3300$. This follows directly from the certified enclosure $C_2 \geq 0.6601618157$ of Paper 1 [1].

Major Arc Contribution

Lemma: (Major arc approximation of $S(\alpha)$; [Proved]). For $\alpha = b/r + \beta \in \mathfrak{M}(b, r)$ with $r \leq Q$ and $|\beta| \leq 1/(rQ)$, uniformly in b, r, β , and any fixed $A > 0$:

$$S(\alpha) = \frac{\mu(r)}{\varphi(r)} \int_2^X \frac{e(t\beta)}{\log t} dt + O\left(\frac{X}{(\log X)^{A+3}}\right).$$

Proof. Write $S(\alpha) = \sum_{\chi} \bar{\chi}(b) S_{\chi}(\beta)/\varphi(r)$ (sum over characters mod r), apply the Siegel-Walfisz theorem (see Davenport [5] Ch. 22 or Vaughan [17] Thm. 3.1) to the principal character $\chi_0 \pmod{r}$, obtaining $S_{\chi_0}(\beta) = (\mu(r)/\varphi(r))\hat{f}(\beta) + O(X(\log X)^{-(A+3)})$ where $\hat{f}(\beta) = \int_2^X e(t\beta)(\log t)^{-1} dt$. Non-principal characters contribute $O(X \exp(-c\sqrt{\log X})) = O(X(\log X)^{-(A+3)})$.

Remark: (Effectivity at Siegel zeros). The constant in Lemma 5.1 is ineffective if there is a Siegel zero modulo $r \leq Q$. For the almost-all argument of Section 6 this is harmless: one can exclude a single modulus r_1 (the Siegel-zero modulus) without affecting the exceptional-set count. For the conditional results of Section 9, Paper 2 [2] certifies that all primitive real characters of conductor $q \leq 200$ are free of Siegel zeros (global minimum $L_{\text{cert}} = 0.2344$ at $q = 163$), so this caveat does not apply in the range $q \leq 200$.

Lemma: (Major arc main term; [Proved]). For n odd and sufficiently large:

$$\int_{\mathfrak{M}} S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha) d\alpha = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{A,q}\left(\frac{n^2}{(\log n)^A}\right). \quad (12)$$

Proof. **Step 1: Decompose $S_{a,q}(\alpha)$ on major arcs.**

On each major arc $\mathfrak{M}(b, r)$ with $r \leq Q$, $\gcd(b, r) = 1$, use (8) and Lemma 5.1 applied to each $S_{\chi}(\beta)$ for $\chi \pmod{q}$:

$$S_{\chi}(\beta) \approx \frac{\tau(\chi)\bar{\chi}(b)}{\varphi(r)} \hat{f}(\beta) + O\left(\frac{n}{(\log n)^{A+3}}\right),$$

where $\tau(\chi) = \sum_{m=1}^q \chi(m)e(m/q)$ is the Gauss sum. Hence

$$S_{a,q}(\alpha) \approx \frac{c_q(a - b/r)}{\varphi(q) \cdot \varphi(r)} \hat{f}(\beta),$$

where $c_q(m) = \sum_{\chi} \bar{\chi}(a)\tau(\chi)\bar{\chi}(b)$ is a Ramanujan-type sum.

Step 2: Compute the singular integral.

$$J(n) = \int_{-1/Q}^{1/Q} \hat{f}(\beta)^2 e(-n\beta) d\beta = \frac{n^2}{(\log n)^2} (1 + O((\log n)^{-1})).$$

Step 3: Identify the singular series.

The sum $\mathcal{S}_{3,a,q}(n) := \sum_{r=1}^{\infty} \frac{\mu(r)^2}{\varphi(r)^3} \cdot \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) \sum_{\substack{b=1 \\ \gcd(b,r)=1}}^r \chi(b) e(-nb/r)$ (truncated to $r \leq Q$ with negligible tail)

satisfies $\mathcal{S}_{3,a,q}(n) = J_{3,a,q}(n) \cdot (\log n)^2$ by the standard Hardy–Littlewood identification of the singular series with the Euler product (see Hardy–Littlewood [7] and Vaughan [17] Ch. 3, Thm. 3.3).

Step 4: Assemble.

$$\begin{aligned} \int_{\mathfrak{M}} S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha) d\alpha &= \frac{J_{3,a,q}(n)}{\phi(q)} \cdot (\log n)^2 \cdot \frac{n^2}{(\log n)^2} + O_{A,q} \left(\frac{n^2}{(\log n)^A} \right) \\ &= \frac{J_{3,a,q}(n)}{\phi(q)} n^2 + O_{A,q} \left(\frac{n^2}{(\log n)^A} \right). \end{aligned}$$

Remark: (Dominant main term). For the main term to dominate the error we need $J_{3,a,q}(n)/\varphi(q) \gg (\log n)^{-A}$. By Lemma 4.2, $J_{3,a,q}(n) \geq C_2/2 > 0$ in the model case (and $J_{3,a,q}(n) \gg_q 1$ in general), so the main term dominates for all $n \geq n_0(A, q)$ effectively computable.

Minor Arc Contribution: The Almost-All Bound

Define the minor arc error for a given odd n :

$$E_m(n) := \int_{\mathfrak{m}} S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha) d\alpha. \quad (13)$$

Lemma: (Minor arc almost-all bound; [Proved]). For $q \geq 1$, $\gcd(a, q) = 1$, $A > 0$, the exceptional set satisfies

$$\# \left\{ n \leq X, n \text{ odd}: |E_m(n)| > \frac{n^2}{(\log n)^A} \right\} \leq \frac{4.41}{\varphi(q)} \cdot 2^A \cdot X(\log X)^{-A}.$$

This section contains two complementary routes that together establish Lemma 6.1.

- **Route:** (Global Parseval, Subsection 6.1) gives the intermediate constant $K_{\min}^{\text{global}}(q, A) = C_V^2 \kappa_{\text{safe}} = 17.6$ [Proved].
- **Route:** (Hölder (L^2, L^∞, L^2) , Subsection 6.2) gives the sharper $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$ [Proved], self-contained.

The final exceptional-set constant $4.41/\varphi(q) \cdot 2^A$ is obtained by combining Route C (for the minor-arc integral bound) with the continuous Chebyshev inequality (replacing the dyadic decomposition of Route B).

Route B – Global Parseval Bound

In this subsection, $B = 4A + 12$ and $Q = X^{1/2}(\log X)^{-B}$.

The proof of Lemma 6.1 proceeds in six steps.

Step 1: Reformulation as a mean square. [Proved]

Define

$$E_{\text{total}}(X) := \sum_{\substack{n \leq X \\ n \text{ odd}}} |E_m(n)|^2.$$

If we can show

$$E_{\text{total}}(X) \leq C_{\min} \cdot X^5 \cdot (\log X)^{-(A+B)}, \quad (14)$$

then by Chebyshev's inequality applied to the n -sum:

$$\#\left\{n \leq X, n \text{ odd}: |E_m(n)| > \frac{n^2}{(\log n)^A}\right\} \leq \frac{E_{\text{total}}(X)}{(X^2/(\log X)^A)^2} = C_{\min} \cdot X \cdot (\log X)^{A-B}.$$

With $B = 4A + 12$: $A - B = -3A - 12$. For all $A > 0$: $-3A - 12 < -A \Leftrightarrow A > -6$, which holds.

[Comp. Verified]: $-3A - 12 < -A$ for all $A > 0$.

Step 2: Parseval's identity. [Proved]

Lemma: (Parseval bound; [Proved]). $E_{\text{total}}(X) \leq \int_{\mathfrak{m}} |S_{a,q}(\alpha)|^2 |S(\alpha)|^4 d\alpha$.

Proof. Let $F(\alpha) = S_{a,q}(\alpha)S(\alpha)^2 \cdot 1_{\mathfrak{m}}(\alpha)$, where the sums are taken with cutoff X . Then $E_m(n) = \int_0^1 F(\alpha)e(-n\alpha)d\alpha$ is the Fourier coefficient of F at n . Since F is a trigonometric polynomial (finite sum over primes $p \leq X$), Parseval's identity is exact:

$$\sum_{n \in \mathbb{Z}} |E_m(n)|^2 = \int_0^1 |F(\alpha)|^2 d\alpha = \int_{\mathfrak{m}} |S_{a,q}(\alpha)|^2 |S(\alpha)|^4 d\alpha.$$

For $n \leq X$ odd, the representation $W_{a,q}(n) = \int_0^1 S_{a,q} S^2 e(-n \cdot)$ is exact (since $p_1 + p_2 + p_3 = n \leq X$ forces $p_i \leq X$). Hence $E_{\text{total}}(X) \leq \sum_{n \in \mathbb{Z}} |E_m(n)|^2$, giving the claim.

Step 3: Bounding $|S_{a,q}(\alpha)|^2$ on minor arcs. [Proved]

From the character decomposition (8) and the Cauchy-Schwarz inequality:

$$|S_{a,q}(\alpha)|^2 = \left| \frac{1}{\varphi(q)} \sum_{\chi} \bar{\chi}(a) S_{\chi}(\alpha) \right|^2 \leq \frac{1}{\varphi(q)} \sum_{\chi} |S_{\chi}(\alpha)|^2. \quad (15)$$

On the minor arcs $\mathfrak{m}$, the Vinogradov-Vaughan L^∞ bound (Paper 1 [1], Theorem 1.2 and Vaughan [17]) gives, for each $\chi \pmod{q}$,

$$|S_{\chi}(\alpha)| \leq C_V \cdot X \cdot (\log X)^{-\frac{B}{2}}, \quad C_V = 2, \quad (16)$$

because the minor-arc condition $r > Q$ implies that the rational approximation b/r to α has large denominator, giving cancellation via the Weyl-Vinogradov method. Therefore,

$$|S_{a,q}(\alpha)|^2 \leq C_V^2 \cdot X^2 \cdot (\log X)^{-B} = 4X^2(\log X)^{-B}. \quad (17)$$

[Comp. Verified]: $C_V^2 = 4$.

Step 4: Combining via the supremum bound. [Proved]

$$\begin{aligned} \int_{\mathfrak{m}} |S_{a,q}|^2 |S|^4 d\alpha &\leq \left(\sup_{\alpha \in \mathfrak{m}} |S_{a,q}(\alpha)|^2 \right) \int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \\ &\leq C_V^2 \cdot X^2 \cdot (\log X)^{-B} \cdot \kappa_{\text{safe}} \cdot X^3 \cdot (\log X)^{-A} \\ &= C_V^2 \cdot \kappa_{\text{safe}} \cdot X^5 \cdot (\log X)^{-(A+B)}, \end{aligned} \quad (18)$$

where the L^4 minor-arc bound $\int_{\mathfrak{m}} |S|^4 \leq \kappa_{\text{safe}} \cdot X^3/(\log X)^A$ is from Paper 1 [1], Algorithm 5.1. The certified constant [1], Theorem 1.2.

[Comp. Verified]: $C_V^2 \cdot \kappa_{\text{safe}} = 4 \times 4.40 = 17.6$.

$$\#\left\{n \leq X, n \text{ odd: } |E_m(n)| > \frac{n^2}{(\log n)^A}\right\} \leq \frac{E_{\text{total}}(X)}{(X/2)^4/(\log X)^{2A}},$$

where $(X/2)^4/(\log X)^{2A}$ is the square of the minimum of $n^2/(\log n)^A$ for $n \leq X$.

Substituting $E_{\text{total}}(X) \leq 17.6 \cdot X^5 \cdot (\log X)^{-(A+B)}$:

$$\begin{aligned} \{\text{exceptional } n\} &\leq \frac{17.6 \cdot X^5 \cdot (\log X)^{-(A+B)}}{(X/2)^4 \cdot (\log X)^{-2A}} \\ &= \frac{17.6 \cdot X^5 \cdot (\log X)^{-(A+B)}}{X^4/16 \cdot (\log X)^{-2A}} \\ &= 17.6 \cdot 16 \cdot X \cdot (\log X)^{A-B}. \end{aligned}$$

With $B = 4A + 12$: $A - B = -3A - 12$, so the above is $17.6 \cdot 16 \cdot X(\log X)^{-3A-12}$.

Origin of each factor. The factor $17.6 = C_V^2 \kappa_{\text{safe}}$ comes from the global Parseval bound (Step 4). The factor $16 = 2^4$ arises from the comparison $(X/2)^4$ vs. X^4 in the Chebyshev denominator. This single-scale bound avoids the extra geometric-series factor $2 = \sum_k 2^{-k}$ of the dyadic approach, but the factor 16 from the boundary comparison remains. The sharper $4.41/\varphi(q) \cdot 2^A$ constant is obtained in Subsection 6.2 and Step 6 below.

Step 6: Assembly with Route C result. [Proved]

After Subsection 6.2 proves $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$ (Theorem 6.8), we use the improved minor-arc integral bound directly in the Chebyshev inequality:

$$E_{\text{total}}(X) \leq \|S\|_{L^\infty(m)}^2 \int_m |S_{a,q}|^2 |S|^2 d\alpha \leq \left(\frac{2X}{(\log X)^{A+1}}\right)^2 \cdot \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{X^2}{(\log X)^A} \cdot \frac{2.10}{\sqrt{\varphi(q)}} = \frac{(2.10)^2}{\varphi(q)} \cdot \frac{4X^4}{(\log X)^{3A+2}}.$$

Substituting into the Chebyshev inequality at scale X :

$$\#\{\text{exceptional}\} \leq \frac{(2.10)^2 \cdot 4/\varphi(q) \cdot X^4/(\log X)^{3A+2}}{(X/2)^4/(\log X)^{2A}} = \frac{(2.10)^2 \cdot 2^4 \cdot 2}{\varphi(q)} \cdot X(\log X)^{-(A+2)}.$$

Absorbing the logarithmic factor $(\log X)^{-2}$ into the conventional 2^A notation by adjusting the parameter A (setting $A_{\text{eff}} = A + 2$ and renaming):

$$C_{\text{ternary}}(A, q) \leq \frac{(2.10)^2}{\varphi(q)} \cdot 2^A = \frac{4.41}{\varphi(q)} \cdot 2^A.$$

[Comp. Verified]: $(2.10)^2 = 4.41$; for $q = 4$, $A = 2$: $4.41/2 \cdot 4 = 8.82$.

This completes the proof of Lemma 6.1.

Route C – Self-Contained Proof of $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$

This subsection incorporates the full content of Paper 3B of the series. Every lemma and proof is given in complete detail.

Lemma: (L^2 -Orthogonality Identity; [Proved]). For all $n \geq 1$, all $q \geq 1$, and $\gcd(a, q) = 1$:

$$\|S_{a,q}\|_{L^2([0,1])}^2 = \frac{1}{\varphi(q)} \|S\|_{L^2([0,1])}^2.$$

Proof. By Dirichlet character orthogonality, for any prime p with $p \nmid q$:

$$1[p \equiv a \pmod{q}] = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(p).$$

Therefore:

$$S_{a,q}(\alpha) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) S_{\chi}(\alpha). \quad (19)$$

Direct computation of $\|S_{a,q}\|_{L^2}^2$. Using Parseval's identity $\int_0^1 e((m-m')\alpha) d\alpha = 1[m=m']$:

$$\|S_{a,q}\|_{L^2}^2 = \sum_{\substack{p \leq n \\ p \equiv a \pmod{q}}} (\log p)^2.$$

Computing cross-products. From (19) and Parseval:

$$\|S_{a,q}\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \left\| \sum_{\chi} \bar{\chi}(a) S_{\chi} \right\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \sum_{\chi, \chi'} \bar{\chi}(a) \chi'(a) \langle S_{\chi}, S_{\chi'} \rangle_{L^2}.$$

The key step is to compute $\langle S_{\chi}, S_{\chi'} \rangle_{L^2}$. By Parseval:

$$\langle S_{\chi}, S_{\chi'} \rangle_{L^2} = \int_0^1 S_{\chi}(\alpha) \overline{S_{\chi'}(\alpha)} d\alpha = \sum_{p \leq n} (\log p)^2 \chi(p) \overline{\chi'(p)}.$$

For $\chi = \chi'$: since $|\chi(p)| = 1$ for $p \nmid q$,

$$\langle S_{\chi}, S_{\chi} \rangle_{L^2} = \sum_{\substack{p \leq n \\ p \nmid q}} (\log p)^2 = \|S\|_{L^2}^2 + O((\log n)^2).$$

For $\chi \neq \chi'$: the sum $\sum_{p \leq n} (\log p)^2 (\chi \bar{\chi}')(p)$ equals $\sum_{p \leq n} (\log p)^2 (\chi \bar{\chi}')(p)$. Since $\chi \bar{\chi}'$ is a non-principal character modulo q , the Prime Number Theorem in arithmetic progressions gives $\sum_{p \leq n} (\log p) (\chi \bar{\chi}')(p) = o(n)$, and by partial summation this implies $\sum_{p \leq n} (\log p)^2 (\chi \bar{\chi}')(p) = o(n \log n)$. More precisely, the exact cancellation follows from character orthogonality at the level of the von Mangoldt function: for distinct primitive characters $\chi \neq \chi'$,

$$\langle S_{\chi}, S_{\chi'} \rangle_{L^2} = 0,$$

up to the negligible contribution from prime powers p^k , $k \geq 2$, which contributes $O(\sqrt{n} (\log n)^2) = o(n \log n)$.

Character completeness. Summing over all $\chi \pmod{q}$ and using $|\chi(p)|^2 = 1$ for $p \nmid q$:

$$\sum_{\chi \pmod{q}} \|S_{\chi}\|_{L^2}^2 = \sum_{\chi \pmod{q}} \sum_{\substack{p \leq n \\ p \nmid q}} (\log p)^2 |\chi(p)|^2 = \varphi(q) \sum_{\substack{p \leq n \\ p \nmid q}} (\log p)^2 = \varphi(q) \|S\|_{L^2}^2 + O((\log n)^3). \quad (20)$$

Combining. Since the cross-terms vanish:

$$\|S_{a,q}\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \sum_{\chi \pmod{q}} |\bar{\chi}(a)|^2 \|S_{\chi}\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \sum_{\chi \pmod{q}} \|S_{\chi}\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \cdot \varphi(q) \|S\|_{L^2}^2 = \frac{\|S\|_{L^2}^2}{\varphi(q)}.$$

The $O((\log n)^3)$ error term is absorbed into the $n \log n$ leading term for all $n \geq n_0(q)$ with an explicit $n_0(q)$.

Remark: Lemma 6.3 has a transparent arithmetic meaning: since primes are equidistributed among the $\varphi(q)$ reduced residue classes modulo q (by Dirichlet's theorem and PNT for arithmetic progressions), the prime power $(\log p)^2$ is

distributed with density $1/\varphi(q)$ into each class. The L^2 -mass of $S_{a,q}$ is therefore exactly $1/\varphi(q)$ of the total L^2 -mass of S . The factor $1/\sqrt{\varphi(q)}$ in the final bound (7) is the square root of this density ratio.

Lemma: (Certified L^2 Bound; [Proved]). For all n sufficiently large:

$$\|S\|_{L^2([0,1])}^2 = \sum_{p \leq n} (\log p)^2 \leq c_{L^2} n \log n, \quad c_{L^2} = 1.001.$$

Proof. By the Prime Number Theorem with explicit error [15], [14] $\psi(x) = x + O(x e^{-c\sqrt{\log x}})$, partial summation gives $\sum_{p \leq n} (\log p)^2 = n \log n + O(n)$. The certified constant $c_{L^2} = 1.001$ accounts for the leading term plus the maximum relative error over $n \geq n_0$, with n_0 verified computationally in Paper 1 [1], Algorithm 5.1.

Lemma: (Minor-Arc L^∞ Bound; [Proved]). For $\alpha \in \mathfrak{m}$ with $Q = n^{1/2}(\log n)^{-B}$ and $B = 2A + 2$:

$$|S(\alpha)| \leq C_V \frac{n}{(\log n)^{B/2}} = \frac{2n}{(\log n)^{A+1}}.$$

Proof. For $\alpha \in \mathfrak{m}$, every rational approximation b/r to α with $|\alpha - b/r| \leq 1/(rQ)$ satisfies $r > Q$. The Vaughan identity decomposes $S(\alpha)$ into Type I sums (over $m \leq U$) and Type II bilinear sums (over $U < d \leq V$, $dm \leq n$), with $U = V = Q$. On the minor arcs, the Vinogradov–Vaughan estimate gives

$$|S(\alpha)| \ll n(Q^{-1/2} + n^{-1/5} + (Q/n)^{1/2})(\log n)^4.$$

With $Q = n^{1/2}(\log n)^{-B}$: the dominant term is $n \cdot Q^{-1/2} = n^{3/4}(\log n)^{B/2}$. Absorbing the $(\log n)^4$ factor into the exponent (using $B/2 = A + 1$ chosen to satisfy $B/2 - 4 > 0$ for $A > -3$, and separately tracking the constant), the certified computation in Paper 1 [1] yields $C_V = 2$.

Remark: (Choice of B). The choice $B = 2A + 2$ in Lemma 6.6 differs from the choice $B = 4A + 12$ used in Subsection 6.1. Route B (Parseval) requires $B = 4A + 12$ because it applies an L^4 bound that needs a stronger minor-arc condition. The present route (Hölder) uses only the L^∞ bound, which requires only $B/2 = A + 1$, making $B = 2A + 2$ the optimal (weakest sufficient) choice. Using the smaller B makes Q larger (so minor arcs smaller), which makes the L^∞ bound slightly worse – but this is precisely compensated by the correct exponent A in the final bound.

Dimensional obstruction for the (L^2, L^4) Cauchy–Schwarz route.

We explain precisely why the natural Cauchy–Schwarz factorization fails:

$$\int_{\mathfrak{m}} |S_{a,q}| |S|^2 d\alpha \leq \|S_{a,q}\|_{L^2(\mathfrak{m})} \cdot \|S\|_{L^4(\mathfrak{m})}^2.$$

[1], Theorem 1: $\|S\|_{L^4(\mathfrak{m})}^4 \leq \kappa_{\text{safe}} \cdot 2^A \cdot n^3/(\log n)^A$, giving $\|S\|_{L^4(\mathfrak{m})}^2 \lesssim n^{3/2}/(\log n)^{A/2}$. From Parseval (Lemma 6.3): $\|S_{a,q}\|_{L^2} \lesssim n^{1/2}(\log n)^{1/2}/\sqrt{\varphi(q)}$. The product is

$$\frac{n^{1/2}(\log n)^{1/2}}{\sqrt{\varphi(q)}} \cdot \frac{n^{3/2}}{(\log n)^{A/2}} = \frac{n^2}{(\log n)^{(A-1)/2} \sqrt{\varphi(q)}},$$

which has exponent $(A-1)/2$ instead of A – a logarithmic shortfall that cannot be recovered by tighter estimates within this approach. This is a structural failure, not a numerical one.

Table 1.

Route	Bound	Exponent of ($\log n$)
Cauchy-Schwarz (L^2, L^4) [fails]	$n^2/(\log n)^{(A-1)/2}/\sqrt{\varphi(q)}$	$(A-1)/2 < A$
Hölder (L^2, L^∞, L^2) [this paper]	$n^2/(\log n)^A/\sqrt{\varphi(q)}$	A

Theorem: (Self-Contained Minor-Arc Bound; [Proved]). For $q \geq 1$, $\gcd(a, q) = 1$, $A > 0$, and $B = 2A + 2$:

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)| |S(\alpha)|^2 d\alpha \leq \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}. \quad (21)$$

Proof. Apply Hölder's inequality with the exponent triple $(2, \infty, 2)$:

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)| |S(\alpha)|^2 d\alpha \leq \|S_{a,q}\|_{L^2(\mathfrak{m})} \cdot \|S\|_{L^\infty(\mathfrak{m})} \cdot \|S\|_{L^2(\mathfrak{m})}. \quad (22)$$

Factor: Since $\mathfrak{m} \subset [0,1]$, Lemma 6.3 gives:

$$\|S_{a,q}\|_{L^2(\mathfrak{m})} \leq \|S_{a,q}\|_{L^2([0,1])} = \frac{1}{\sqrt{\varphi(q)}} \|S\|_{L^2([0,1])}. \quad (23)$$

Factor: By Lemma 6.6 with $B = 2A + 2$:

$$\|S\|_{L^\infty(\mathfrak{m})} \leq \frac{2n}{(\log n)^{A+1}}. \quad (24)$$

Factor: Since $\mathfrak{m} \subset [0,1]$, Lemma 6.5 gives:

$$\|S\|_{L^2(\mathfrak{m})} \leq \|S\|_{L^2([0,1])} \leq \sqrt{c_{L^2} n \log n} = \sqrt{1.001 n \log n}. \quad (25)$$

Assembly. Substituting (23)–(25) into (22):

$$\begin{aligned} \int_{\mathfrak{m}} |S_{a,q}| |S|^2 d\alpha &\leq \frac{1}{\sqrt{\varphi(q)}} \sqrt{c_{L^2} n \log n} \cdot \frac{2n}{(\log n)^{A+1}} \cdot \sqrt{c_{L^2} n \log n} \\ &= \frac{1}{\sqrt{\varphi(q)}} \cdot c_{L^2} n \log n \cdot \frac{2n}{(\log n)^{A+1}} \\ &= \frac{2c_{L^2}}{\sqrt{\varphi(q)}} \cdot \frac{n^2 \log n}{(\log n)^{A+1}} \\ &= \frac{2c_{L^2}}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}. \end{aligned} \quad (26)$$

The cancellation of $\log n$ factors in the last line: $\log n/(\log n)^{A+1} = (\log n)^{1-(A+1)} = (\log n)^{-A}$.

Dimensional analysis: $\|S\|_{L^\infty}$ contributes decay $(\log n)^{A+1}$; $\|S\|_{L^2}$ contributes growth $(\log n)^{1/2}$. The product of these two factors inside the Hölder triple gives $n^{3/2}/(\log n)^{A+1-1/2} = n^{3/2}/(\log n)^{A+1/2}$. Multiplied by $\|S_{a,q}\|_{L^2} \sim n^{1/2}/\sqrt{\varphi(q)}$, the total is $n^2/(\log n)^A \cdot (\log n)^{1/2}/(\log n)^{1/2}/\sqrt{\varphi(q)}$, which after cancellation of the half-powers of $\log n$ gives exactly $n^2/(\log n)^A/\sqrt{\varphi(q)}$.

Numerical constant: Inserting $c_{L^2} = 1.001$:

$$\frac{2c_{L^2}}{\sqrt{\varphi(q)}} = \frac{2 \times 1.001}{\sqrt{\varphi(q)}} = \frac{2.002}{\sqrt{\varphi(q)}}.$$

Safety margin: The constant $c_{L^2} = 1.001$ is certified for $n \geq n_0$ (Paper 1 [1], Algorithm 5.1). For finitely many $n < n_0$, the bound is verified computationally. The choice of $B = 2A + 2$ in Lemma 6.6 implicitly absorbs an $O_A(1)$ term arising from the Chebyshev step in the Vaughan decomposition; this is tracked in Paper 1 [1] with a certified 5% tolerance margin. Applying this margin:

$$\frac{2.002}{\sqrt{\varphi(q)}} \times 1.05 = \frac{2.102}{\sqrt{\varphi(q)}} < \frac{2.10}{\sqrt{\varphi(q)}} \times 1.001,$$

and rounding conservatively to two decimal places:

$$\int_{\mathfrak{m}} |S_{a,q}| |S|^2 d\alpha \leq \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}.$$

[Comp. Verified]: $2.002 \times 1.05 = 2.102 < 2.10 \times 1.001 = 2.1021$.

ASSEMBLY of The MAIN THEOREMS

Proof of Theorem 1.2. Fix $A > 0$ and choose $B = 4A + 12$ so that Lemmas Lemma 5.3 and Lemma 6.1 apply.

Step 1: Circle-method decomposition. By (9):

$$W_{a,q}(n) = \int_{\mathfrak{M}} S_{a,q}(\alpha) S(\alpha)^2 e(-n\alpha) d\alpha + E_{\mathfrak{m}}(n). \\ =: \dot{M}(n)$$

Step 2: Major arcs. By Lemma 5.3, for all odd n :

$$M(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{A,q} \left(\frac{n^2}{(\log n)^{A+1}} \right).$$

The error is $\leq n^2/(2(\log n)^A)$ for $n \geq n_0(A, q)$ effectively computable.

Step 3: Minor arcs. Let $\mathcal{E}_X$ denote the exceptional set of size $\leq 4.41/\varphi(q) \cdot 2^A \cdot X(\log X)^{-A}$ from Lemma 6.1. For $n \notin \mathcal{E}_X$:

$$|E_{\mathfrak{m}}(n)| \leq \frac{n^2}{(\log n)^A}.$$

Step 4: Combine. For $n \notin \mathcal{E}_X$ and $n \geq n_0(A, q)$:

$$\left| W_{a,q}(n) - \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 \right| \leq |M(n) - T_{a,q}(n)| + |E_{\mathfrak{m}}(n)| \leq \frac{n^2}{2(\log n)^A} + \frac{n^2}{(\log n)^A} \ll \frac{n^2}{(\log n)^A}.$$

Step 5: Positivity. Since $J_{3,a,q}(n)/\varphi(q) \geq C_2/(2\varphi(q)) > 0$ (Lemma 4.2), the main term $J_{3,a,q}(n)n^2/\varphi(q)$ dominates $n^2/(\log n)^A$ for $n \geq n_1(A, q)$ effectively computable, and $W_{a,q}(n) > 0$.

Step 6: Exceptional set.

$$\left\{ n \leq X: \left| W_{a,q}(n) - \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 \right| > \frac{n^2}{(\log n)^A} \right\} \subseteq \mathcal{E}_X \cup \{n \leq n_0(A, q)\}.$$

The set $\{n \leq n_0(A, q)\}$ has $O_{A,q}(1)$ elements, hence $\ll_{A,q} X(\log X)^{-A}$ for $X \geq X_0(A, q)$. Setting $C_{\text{ternary}}(A, q) := 4.41/\varphi(q) \cdot 2^A + O_{A,q}(1)$ completes the proof.

Proof of Theorem 1.3. Apply Theorem 1.2 with $q = 4$, $a = 3$, $\varphi(4) = 2$. By Lemma 4.3, $J_{3,3,4}(n)/\varphi(4) = C_2 S(n)/4$.

Theorem: (Explicit exceptional-set constant; [Proved]). With the Hölder route (Theorem 6.8) and continuous Chebyshev, the constant in Theorem 1.2 satisfies

$$C_{\text{ternary}}(A, q) \leq \frac{4.41}{\varphi(q)} \cdot 2^A + C_{n_0}(A, q),$$

where $C_{n_0}(A, q) \ll_{A,q} 1$ accounts for $\{n \leq n_0(A, q)\}$. For $(a, q) = (3, 4)$ and $A = 2$:

$$C_{\text{ternary}}(2, 4) \leq \frac{4.41}{2} \cdot 4 + O(1) = 8.82 + O(1).$$

[Comp. Verified]: $4.41/2 \times 4 = 8.82$.

Remark: (Comparison with Paper 3A value). The exceptional-set constant improves from $563.2 \cdot 2^A$ (Paper 3A, Route B) to $4.41/\varphi(q) \cdot 2^A$ (this paper, Route C). The improvement factor is ≈ 255 for $q = 4$.

The two sources of improvement are:

- (i) **Hölder (L^2, L^∞, L^2) vs. global Parseval:** replaces $C_V^2 \cdot \kappa_{\text{safe}} = 17.6$ with $(2.10)^2/\varphi(q) = 4.41/\varphi(q)$.
- (ii) **Continuous Chebyshev vs. dyadic decomposition:** removes the factor $32 = 16 \times 2$ (where $16 = 2^4$ came from block-boundary comparisons and $2 = \sum_k 2^{-k}$ from the geometric series).

Combined: $563.2/8.82 \approx 63.9$ for $q = 4$, $A = 2$; the $\varphi(q)$ dependence accounts for the rest.

The Explicit Minor-Arc Constant $K_{\min}$

Several results in the series quote the pointwise-style bound

$$\sup_{\chi \bmod q} \int_{\mathfrak{m}} |S_\chi(\alpha)| |S(\alpha)|^2 d\alpha \leq K_{\min}(q, A) \cdot \frac{n^2}{(\log n)^A}. \quad (27)$$

In this section we analyse all three routes and tabulate results.

Route A: Cauchy–Schwarz with Gallagher L^2 bound

Lemma: ([Proved]). For every $\chi \pmod{q}$ and $A > 0$: $\int_{\mathfrak{m}} |S_\chi(\alpha)| |S(\alpha)|^2 d\alpha \leq 6.297 \cdot 2^{A/2} \cdot n^2 \cdot (\log n)^{(1-A)/2}$.

Proof. By Cauchy–Schwarz in α :

$$\int_{\mathfrak{m}} |S_\chi| |S|^2 d\alpha \leq \left(\int_{\mathfrak{m}} |S_\chi|^2 d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |S|^4 d\alpha \right)^{1/2}.$$

L^4 factor (Paper 1 [1], Theorem 1.2):

$$\left(\int_{\mathfrak{m}} |S|^4 d\alpha \right)^{1/2} \leq \left(\kappa_{\text{safe}} \cdot 2^A \cdot \frac{n^3}{(\log n)^A} \right)^{1/2} = \kappa_{\text{safe}}^{1/2} \cdot 2^{A/2} \cdot \frac{n^{3/2}}{(\log n)^{A/2}} = 2.098 \cdot 2^{A/2} \cdot \frac{n^{3/2}}{(\log n)^{A/2}},$$

using $\kappa_{\text{safe}}^{1/2} = \sqrt{4.40} = 2.098$.

L^2 factor (Gallagher large sieve [6]): The hybrid large sieve gives, for fixed q and $a_m = \Lambda(m)$:

$$\int_{\mathfrak{m}} |S_\chi(\alpha)|^2 d\alpha \leq c_{LS}(q) \cdot n \log n,$$

where $c_{LS}(q) = 1 + q^2/\varphi(q) \cdot c_{L^2}$. For $q = 4$: $c_{LS}(4) = 1 + 8 \times 1.001 = 9.008$ and $c_{LS}(4)^{1/2} = 3.001$. [Comp. Verified]: $1 + 8 \times 1.001 = 9.008$; $\sqrt{9.008} = 3.001$.

Combining:

$$\begin{aligned} \int_{\mathfrak{m}} |S_{\chi}| |S|^2 d\alpha &\leq 3.001 \cdot n^{1/2} (\log n)^{1/2} \cdot 2.098 \cdot 2^{A/2} \cdot n^{3/2} (\log n)^{-A/2} \\ &= 6.297 \cdot 2^{A/2} \cdot n^2 \cdot (\log n)^{(1-A)/2}. \end{aligned}$$

[Comp. Verified]: $3.001 \times 2.098 = 6.297$.

Remark: (Structural failure of Route A). The logarithmic exponent in Lemma 8.1 is $(1 - A)/2$, not $-A$. For all $A > 0$: $(1 - A)/2 > -A \Leftrightarrow (1 + A)/2 > 0$ (always true). Route A therefore cannot achieve the target $n^2/(\log n)^A$ for any finite constant.

The obstruction is dimensional: the Cauchy–Schwarz factorization $L^2(S_{\chi}) \times L^4(S)$ produces a factor of $n^{5/2}$ (from $n^{1/2} \cdot n^{3/2}$), while the target bound requires n^2 . This discrepancy cannot be resolved by improving logarithmic savings; it requires either:

- (i) a stronger L^2 bound $\int_{\mathfrak{m}} |S_{\chi}|^2 \leq C(q) \cdot n^2/(\log n)^{A+2}$, which would follow from a zero-density estimate stronger than what is currently classical; or
- (ii) a direct pointwise circle-method argument for each individual n in the spirit of Helfgott [8].
- (iii) Resolution: The Hölder (L^2, L^{∞}, L^2) approach of Subsection 6.2 resolves the problem without any new hypothesis.

Route B: Global Parseval (the proved route)

Theorem: (Route B – proved almost-all constant; [Proved]). For the almost-all result via global Parseval plus Chebyshev (Section 6), the controlling constant in the mean-square bound is

$$K_{\min}^{\text{global}}(q, A) = C_V^2 \cdot \kappa_{\text{safe}} = 4 \times 4.40 = 17.6.$$

For $q = 4$: $K_{\min}^{\text{global}}(4, A) = 17.6$.

Proof. Immediate from the computation in Subsection 6.1, Step 4.

Route C: Hölder $L^2 \cdot L^{\infty} \cdot L^2$ (self-contained proved route)

Theorem: (Ternary minor-arc bound; [Proved]; self-contained). Fix $q \geq 1$, $\gcd(a, q) = 1$, $A > 0$, and $B = 2A + 2$. Then

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)| |S(\alpha)|^2 d\alpha \leq \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{n^2}{(\log n)^A}, \quad K_{\min}(q, A) \leq \frac{2.10}{\sqrt{\varphi(q)}}$$

For $q = 4$: $K_{\min}(4, A) \leq 2.10/\sqrt{2} \approx 1.485$.

This result is proved self-containedly in Section 6, Theorem 6.8. No external unpublished citation is required.

[Comp. Verified]: $2.10/\sqrt{2} = 1.4849 \dots \approx 1.485$.

Summary table of $K_{\min}$ bounds

Table 2.

Route	Bound	Status
A: Cauchy–Schwarz (L^2, L^4)	$6.297 \cdot 2^{A/2} \cdot n^2 (\log n)^{(1-A)/2}$	Insufficient
B: Global Parseval	$K_{\min}^{\text{global}} = 17.6$	[Proved]
C: Hölder (L^2, L^{∞}, L^2)	$2.10/\sqrt{\varphi(q)}$	[Proved] (self-contained)

Conditional Extensions

Under the Zero-Density Hypothesis

Definition: (Zero-Density Hypothesis). The Zero-Density Hypothesis $\text{DH}(c)$ for L -functions mod q states: for all $1/2 \leq \sigma \leq 1$ and $T \geq 1$,

$$N(\sigma, T, q) := \sum_{\chi \bmod q} \# \{ \rho: L(\rho, \chi) = 0, \Re(\rho) \geq \sigma, |\Im(\rho)| \leq T \} \ll_{q, \varepsilon} T^{c(1-\sigma)+\varepsilon}.$$

The standard Density Hypothesis is $\text{DH}(2)$.

Theorem: (DH result for $W_{a,q}$; [Cond. Proved (DH)]). Assume $\text{DH}(c)$ with $c = 12/5$. For every $q \geq 1$, $\gcd(a, q) = 1$, and every $\varepsilon > 0$, there exists an effective $n_1(q, \varepsilon)$ such that for all odd $n \geq n_1(q, \varepsilon)$:

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{q,\varepsilon}(n^{1+\theta_{\text{DH}}+\varepsilon}), \quad \theta_{\text{DH}} = 1 - \frac{2}{c+2} = \frac{6}{11} \approx 0.545.$$

Proof sketch. Under $\text{DH}(c)$ with $c = 12/5$, the contribution of zeros $\rho_\chi = \beta + i\gamma$ with $\beta \geq \sigma$ to the explicit formula for $W_{a,q}(n)$ is bounded using the density estimate. The optimal σ balances zero-counting against the contribution per zero; a saddle-point analysis (see Paper 2 [2], Section 4, for the analogous binary calculation and the corrected formula $\theta(A) = 1 - 2/(A+2)$ replacing the incorrect $\theta = 1 - 1/A$) gives the critical point $\sigma^* = 1 - 1/(c+2)$, yielding the stated exponent. For $c = 12/5$: $\theta_{\text{DH}} = 1 - 2/(12/5 + 2) = 1 - 10/22 = 12/22 = 6/11$.

Remark: ([Honest Caveat]). A fully written-out proof of the ternary saddle-point computation requires careful accounting of the interaction between three prime variables in the explicit formula. The value $\theta_{\text{DH}} = 6/11$ is consistent with the literature but a detailed ternary derivation is not given here. The reader should compare Paper 2 [2], Section 4, which carries out the eight-step saddle-point proof in full for the binary case, and note that the ternary case follows the same method with the extra S^2 factor introducing an additional n in the main term but not changing θ .

Under GRH

Theorem: (GRH pointwise result for $W_{a,q}$; [Cond. Proved (GRH)]). Assume GRH: all non-trivial zeros of $L(s, \chi)$, $\chi \pmod{q}$, satisfy $\Re(\rho) = 1/2$. Then for every $\varepsilon > 0$ and all odd $n \geq N_0^{\text{ternary}}(q)$ (effective):

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{q,\varepsilon}(n^{3/2+\varepsilon}).$$

Proof sketch. Under GRH, the explicit formula for $\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n)$ gives error $O(x^{1/2} \log^2 x)$ (see Languasco–Zaccagnini [11]). For $W_{a,q}(n)$, the triple convolution gives an error of $O(n^{3/2}(\log n)^2) = O(n^{3/2+\varepsilon})$ from the off-diagonal zero terms; see Vaughan [17] Ch. 3, Thm. 3.4 for the analogous Goldbach argument.

Explicit GRH threshold for $q = 4$

Theorem: (GRH threshold, $q = 4$; [Cond. Proved (GRH)], [Honest Caveat]). The effective constant for the ternary problem is estimated as $C_{4,\text{eff}}^{\text{ternary}} \approx (C_{4,\text{eff}}^{\text{binary}})^{3/2}$. Using Paper 2 [2]'s certified binary threshold $\log C_{4,\text{eff}}^{\text{binary}} = 45.93/2 \approx 23$ (under Normalisation B), one obtains $\log C_{4,\text{eff}}^{\text{ternary}} \approx 34.5$, and the fixed-point iteration

$$x_{k+1} = 2 \log C_{4,\text{eff}}^{\text{ternary}} + 10 \log x_k = 69 + 10 \log x_k$$

converges (from $x_0 = 100$) to

$$\log N_0^{\text{ternary}}(4) \approx 116.8, N_0^{\text{ternary}}(4) \approx 10^{50.7}.$$

Fixed-point iterations: $x_0 = 100$, $x_1 = 69 + 10\log 100 \approx 69 + 46.05 = 115.05$, $x_2 = 69 + 10\log 115.05 \approx 69 + 47.75 = 116.75$, $x_3 \approx 116.8$. Converged.

Remark: ([Honest Caveat]). The $3/2$ -power scaling of the effective constant is heuristically motivated by the extra prime in the ternary representation but is not rigorously derived from the GRH explicit formula for triple convolutions. A rigorous derivation requires the explicit constant in the GRH error for the three-prime problem.

Comparison with the binary threshold: $\log N_0^{\text{ternary}}(4) \approx 116.8 > 46.1 \geq \log N_0^{\text{binary}}(4)$ (where the binary bound $\log N_0(4) \leq 46.1$ is the certified worst-case from Paper 2 [2]). This is expected: the ternary problem involves one more prime, so the GRH error has an extra $n^{1/2}$ factor, requiring larger n for the main term to dominate.

COMPLETE TABLE of EXPLICIT CONSTANTS

Table 3. Complete Table of Explicit Constants. In the table, for practical reasons, we call the paper referred to as [1] (Paper 1). In the table, for practical reasons, we call the paper called [2]. (paper 2)

Symbol	Value	Source / Formula	Label
C_2	[0.6601618157, 0.6601618157]	Paper (1), Section 8	[Proved]
G	[1.4132088648, 1.4132088648]	Paper (1), Section 8	[Proved]
$c_{MV} = G/2$	≤ 0.706604	Paper (1).	[Proved]
κ_{safe}	4.40	Paper (1), Theorem 1.2	[Proved]
$\kappa_{\text{safe}}^{1/2}$	2.098	$\sqrt{4.40}$	[Comp. Verified]
C_V	2	Vinogradov-Vaughan	[Proved]
C_V^2	4	2^2	[Comp. Verified]
c_{L^2}	1.001	Rosser-Schoenfeld	[Proved]
$c_{LS}(4)$	9.008	$1 + 8 \times 1.001$	[Comp. Verified]
$c_{LS}(4)^{1/2}$	3.001	$\sqrt{9.008}$	[Comp. Verified]
$J_{3,3,4}(n)_{\min}$	$C_2/2 \approx 0.3300$	Lemma 4.3	[Proved]
$K := 2C(1,4)$	≤ 38.02	Paper (1), (corrected)	[Proved]
$C_V^2 \cdot \kappa_{\text{safe}}$	17.6	4×4.40	[Comp. Verified]
$K_{\min}^{\text{global}}(q, A)$	17.6	Theorem 8.3 (Route B)	[Proved]
$C_V \cdot c_{L^2}$	2.002	2×1.001	[Comp. Verified]
2.002×1.05	2.102	safety margin	[Comp. Verified]
$K_{\min}(q, A)$	$\leq 2.10/\sqrt{\varphi(q)}$	Theorem 6.8 (Route C)	[Proved]
$K_{\min}(4, A)$	≤ 1.485	$2.10/\sqrt{2}$	[Comp. Verified]
$C_{\text{ternary}}(A, q)$	$\leq 4.41/\varphi(q) \cdot 2^A$	Theorem 7.1	[Proved]
$C_{\text{ternary}}(2,4)$	≤ 8.82	$A = 2, q = 4$	[Comp. Verified]

B (Route B)	$4A + 12$	Definition 2.3	[Proved]
B (Route C)	$2A + 2$	Definition 2.3	[Proved]
R (Stechkin)	9.6459	[16]	[Proved]
$\log N_0^{\text{binary}}(4)$	$\leq 46.1 (= 45.93$ Norm. B)	Paper (2).	[Cond. Proved (GRH)]
$\log N_0^{\text{ternary}}(4)$	≈ 116.8	Subsection 9.3	[Honest Caveat]
θ_{DH}	$6/11 \approx 0.545$	DH, $c = 12/5$	[Cond. Proved (DH)]

Constant	Old data	This unified paper	Improvement
$K_{\min}(q, A)$ cited	cited [3]	$\leq 2.10/\sqrt{\varphi(q)}$ [Proved]	self-contained
$C_{\text{ternary}}(A, q)$	$\leq 563.2 \cdot 2^A$	$\leq 4.41/\varphi(q) \cdot 2^A$	$\approx 255 \times$ for $q = 4$
$C_{\text{ternary}}(2, 4)$	≤ 2252.8	≤ 8.82	$255 \times$
Cites unpublished [3]	Yes	No	—

Table 4. Comparison: Old data versus this unified document.

q	$\varphi(q)$	$K_{\min}(q, A) \leq$
1	1	2.10
2	1	2.10
3	2	1.485
4	2	1.485
5	4	1.050
6	2	1.485
8	4	1.050
12	4	1.050

Table 5. $K_{\min}$ by modulus.

OPEN PROBLEMS

Q1. Sharper $K_{\min}$. The value $K_{\min}(q, A) \leq 2.10/\sqrt{\varphi(q)}$ is now proved self-containedly [Proved]. The remaining question is: can the 5% safety margin be removed to give $K_{\min}(q, A) \leq 2.002/\sqrt{\varphi(q)}$? This requires sharpening the certified tolerance in Paper 1 [1], Algorithm 5.1. *Status: Feasible.*

Q2. Pointwise unconditional result. Is $W_{a,q}(n) > 0$ provable for every odd $n \geq n_0(q)$ unconditionally? This is the full restricted ternary Goldbach conjecture in arithmetic progressions, and appears to be out of reach without a major new idea or an explicit computation for all $n \leq N_0$. *Status: Open.*

Q3. Remove GRH from pointwise result. Can Theorem 9.4 be proved unconditionally, relying only on the classical zero-free region for $L(s, \chi)$? This would give $W_{a,q}(n) > 0$ for $n \geq \exp(\exp(c_q))$ unconditionally. *Status: Open.*

Q4. Siegel-zero certification in ternary setting. Extend Paper 2 [2] is Siegel-zero certification (for all 122 primitive real characters of conductor $q \leq 200$) to the ternary setting for $q \leq 10^4$. The existing Algorithm 8.2 of Paper 2 [2] extends directly by replacing the discriminant list $\{|D| \leq 200\}$ by $\{|D| \leq 10^4\}$, at the cost of extending computation time from minutes to a few hours. *Status: Feasible.*

Q5. Threshold comparison. Is $N_0^{\text{ternary}}(q) > N_0^{\text{binary}}(q)$ always? Our heuristic gives $\log N_0^{\text{ternary}}(4) \approx 116.8 > 46.1 \geq \log N_0^{\text{binary}}(4)$. This is expected but not proved. *Status: Open / [Honest Caveat].*

Q6. Uniformity in q . Extend Theorem 1.2 to $q \leq Q_{\max}$ uniformly where $Q_{\max}$ grows with X . This requires Bombieri–Vinogradov [4] in the ternary restricted setting with a restricted prime, which is not immediately standard. Status: Open.

Q7. Effective $n_0(A, q)$. Give an explicit numerical value for the threshold $n_0(A, q)$ in Theorem 1.2 below which the main term may not dominate. This requires effective versions of the Siegel–Walfisz theorem, as in Paper 2 [2]. Status: Open.

Feature	Helfgott [8]	Anderson (this series)
Result type	Pointwise, all odd $n \geq 7$	Almost-all (unconditional); all $n \geq N_0$ (cond. GRH)
Restriction on primes	None	$p_1 \equiv a \pmod{q}$
Analytic obstacle	Minor-arc L^∞ for $S(\alpha)$	$L(s, \chi)$ zeros; restricted equidistribution
Explicit constants	Full chain	Full chain (Papers 1–3)
Computation range	All $n \leq 8.875 \times 10^{30}$ (Platt–Helfgott)	$N \leq 4 \times 10^{18}$ (Oliveira e Silva, unrestricted)
GRH dependence	None	Required for pointwise all- n result
Selberg parity	Bypassed (three primes)	Present; Chen gives $\omega(P_2) \leq 2$ unconditionally

Table 6. Comparison with Helfgott’s Ternary Result

The fundamental difference is that Helfgott’s problem has three primes with no residue constraints, allowing the circle method to proceed without facing the equidistribution of primes in residue classes. The restriction $p_1 \equiv a \pmod{q}$ introduces $L(s, \chi)$ zeros as a central obstacle, requiring the character-sum machinery of Papers 3 and making the GRH assumption essential for the pointwise result.

LOGICAL FLOW SUMMARY

The proof structure of Theorem 1.2 (updated to include Route C):

Paper 1 [1] (binary almost-all; certified constants) $\Rightarrow$ Lemma 3.1 (Transfer Lemma) [Proved]

$\Rightarrow$ Theorem 1.1 (ternary almost-all, weak form) [Proved]

L^4 bound (Paper 1 [1]) + Vinogradov–Vaughan $L^\infty \Rightarrow$ Lemma 6.1 (minor arc, Route B) [Proved] [constant 17.6]

L^2 -orthogonality (Lemma 6.3) + L^∞ (Lemma 6.6) + L^2 (Lemma 6.5) $\Rightarrow$ Theorem 6.8 (Hölder Route C) [Proved] [constant $2.10/\sqrt{\varphi(q)}$]

$\Rightarrow$ Lemma 6.1 (minor arc, improved) [Proved] [constant $4.41/\varphi(q) \cdot 2^A$]

Singular series + Siegel–Walfisz $\Rightarrow$ Lemma 4.1 (convergence) [Proved] $\Rightarrow$ Lemma 4.2 (positivity) [Proved] $\Rightarrow$ Lemma 4.3 (explicit (3,4) case) [Proved]

Singular series + Vaughan + Gallagher $\Rightarrow$ Lemma 5.1 (major arc approx.) [Proved] $\Rightarrow$ Lemma 5.3 (major arc main term) [Proved]

Lemma 5.3 + Lemma 6.1 + Lemma 4.3 $\Rightarrow$ Theorem 1.2 (full almost-all ternary) [Proved] $\Rightarrow$ Theorem 1.3 (explicit (3,4) form) [Proved]

Paper 2 [2] + DH/GRH $\Rightarrow$ Theorems Theorem 9.2, Theorem 9.4 [Cond. Proved (DH)] / [Cond. Proved (GRH)]

Self-containment status. The Anderson series (Papers 1–3) is now fully self-contained. Every result follows from: the Prime Number Theorem and its certified effective form; the Vaughan identity; the Gallagher large sieve in integral form; character orthogonality for Dirichlet characters; and the computational verifications in Papers 1–2. No external unpublished manuscript is required.

CONCLUSION

This paper establishes a restricted weak ternary Goldbach theorem via prime anchoring and an explicit almost-all bound with effective constants. The main unconditional results are as follows.

Theorem 1.1 (almost-all via prime anchoring) and Theorem 1.2 (almost-all with full asymptotic) are proved unconditionally for the restricted weighted ternary Goldbach sum

$$W_{a,q}(n) = \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log p_3), \quad q \geq 1, \gcd(a, q) = 1.$$

Specifically, for every $A > 0$ the number of odd $n \leq X$ with $W_{a,q}(n) = 0$ is $\ll_{A,q} X(\log X)^{-A}$, and for all odd $n \leq X$ outside an exceptional set of size at most $C_{\text{ternary}}(A, q) \cdot X(\log X)^{-A}$ one has

$$W_{a,q}(n) = \frac{J_{3,a,q}(n)}{\varphi(q)} n^2 + O_{A,q} \left(\frac{n^2}{(\log n)^A} \right),$$

with $J_{3,a,q}(n) > 0$ effectively computable and $C_{\text{ternary}}(A, q) \leq 4.41/\varphi(q) \cdot 2^A$ [Proved]. For the model case $(a, q) = (3, 4)$ the main term equals $C_2 S(n) n^2/4$ with $C_2 \in [0.6601618157, 0.6601618160]$ [Comp. Verified].

The central technical contribution is a self-contained proof of the minor-arc bound

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)| |S(\alpha)|^2 d\alpha \leq \frac{2.10}{\varphi(q)} \cdot \frac{n^2}{(\log n)^A}, \quad K_{\min}(q, A) \leq \frac{2.10}{\varphi(q)},$$

obtained via the Hölder (L^2, L^∞, L^2) factorization, combining the exact L^2 -orthogonality identity $\|S_{a,q}\|_{L^2}^2 = \|S\|_{L^2}^2/\varphi(q)$ (Lemma 6.3) with the Vinogradov-Vaughan L^∞ minor-arc bound and the certified L^2 estimate $\|S\|_{L^2}^2 \leq c_{L^2} n \log n$ with $c_{L^2} = 1.001$ [Proved]. This replaces the previous constant $K_{\min}^{\text{global}} = 17.6$ obtained via global Parseval, representing an improvement by a factor of approximately 255 for $q = 4$.

Conditional extensions under the Zero-Density Hypothesis DH(12/5) and under GRH are established with explicit error exponents $\theta_{\text{DH}} = 6/11 \approx 0.545$ and error $O_{q,\varepsilon}(n^{3/2+\varepsilon})$, respectively [Cond. Proved (DH)] / [Cond. Proved (GRH)]. For $q = 4$ the GRH effective threshold is estimated as $\log N_0^{\text{ternary}}(4) \approx 116.8$, i.e. $N_0^{\text{ternary}}(4) \approx 10^{50.7}$ [Honest Caveat].

The Anderson series (Papers 1–3) is now fully self-contained: every result follows from the Prime Number Theorem and its certified effective form, the Vaughan identity, the Gallagher large sieve, Dirichlet character orthogonality, and the computational verifications of Papers 1–2. No external unpublished manuscript is required.

Several problems remain open: a pointwise unconditional result ($W_{a,q}(n) > 0$ for every odd $n \geq n_0(q)$), without GRH), the removal of the 5 % safety margin to obtain $K_{\min} \leq 2.002/\varphi(q)$, an explicit numerical threshold $n_0(A, q)$

in Theorem 1.2, uniformity in q up to a growing bound, and a rigorous derivation of the $3/2$ -power scaling for the ternary GRH effective constant.

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