

Partial Differential Equations in Economics: A Comprehensive Review

Weiguang Huang *

DrHuang Pty Ltd, Kensington, Sydney, NSW 2033, Australia

ABSTRACT

Partial differential equations (PDEs) provide a common mathematical language for economic dynamics. They arise whenever optimising agents move continuously through time, whenever a population of agents is described by a probability distribution, and whenever economic activity spreads across space. This review synthesises the mathematical foundations, economic interpretations, and computational developments of PDEs used in economic theory and quantitative finance, with coverage extending to 2026. We follow a reproducible review protocol and classify 100 verified primary sources into five streams: optimal control and Hamilton–Jacobi–Bellman (HJB) equations; Kolmogorov–Fokker–Planck (FP) equations for distributional dynamics; PDEs in financial mathematics; mean-field games (MFG) and mean-field control; and spatial, environmental, fractional, and numerical methods. We examine the coupled HJB–FP structure that underlies mean-field models, the nonlinear extensions of Black–Scholes–Merton theory, the continuous-time heterogeneous-agent models that now dominate modern macroeconomics, and the reaction–diffusion and transport equations used in economic geography. Then we assess scientific machine learning, from physics-informed neural networks to deep backward stochastic differential equation solvers, and the fractional operators that capture memory and anomalous diffusion. Our critical analysis identifies persistent weaknesses: weak quantitative validation, the curse of dimensionality, fragile parameter identification, and limited uncertainty quantification. Therefore, we call on researchers to prioritise four concrete directions: hybrid physics-learning solvers that embed conservation and no-arbitrage constraints; structure-preserving high-dimensional discretisations for coupled HJB–Fokker–Planck systems; empirically grounded fractional and nonlocal models whose exponents are tied to measured mechanisms; and shared, open benchmarks that make validation and uncertainty quantification routine. These directions offer the clearest route from elegant illustration to predictive, policy-relevant PDE modelling.

Keywords: Partial differential equations; Hamilton–Jacobi–Bellman equation; Fokker–Planck equation; Black–Scholes; Mean-field games; Heterogeneous agents; Optimal control; Economic geography; Fractional calculus; Scientific machine learning

INTRODUCTION

Economic systems evolve continuously in time, respond to random shocks, and involve the interaction of many agents, so it is not surprising that partial differential equations lie at the core of modern economic theory. A dynamic programming argument turns an intertemporal optimisation problem into a Hamilton–Jacobi–Bellman equation, a continuity argument turns the law of motion of a cross-section of agents into a Kolmogorov or Fokker–Planck equation, and a no-arbitrage argument turns the replication of a contingent claim into a parabolic PDE of Black–Scholes–Merton type. These three mechanisms, optimisation, conservation, and hedging, generate almost every PDE that appears in economics.

The classical foundations were laid in the first half of the twentieth century. Ramsey framed optimal saving as a variational problem [1], Hotelling analysed the extraction of exhaustible resources [2] and the spatial competition of firms, and Koopmans and Cass completed the modern optimal-growth model [3,4]. The stochastic turn followed with Samuelson's discrete-time portfolio problem and Merton's continuous-time treatment, which produced the first economically transparent HJB equation in finance [5]. In parallel, the theory of viscosity solutions gave a rigorous meaning to these fully nonlinear equations [6,7]. Since then the field has expanded in several directions at once: nonlinear and high-dimensional PDEs in finance, distributional dynamics in macroeconomics, mean-field interactions among large populations, spatial diffusion in economic geography, and fractional operators that encode memory.

*Correspondence to: Weiguang Huang, DrHuang Pty Ltd, Kensington, Sydney, NSW 2033, Australia. E-mail: drhuangau@yahoo.com.au

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This review provides a comprehensive and critical synthesis of that literature, covering work published up to 2026. Our objectives are fourfold. First, we clarify the mathematical structure of the canonical PDEs and the economic assumptions that generate them. Second, we compare the numerical methods used to solve them, with attention to stability, accuracy, and dimensionality. Third, we survey applications across finance, macroeconomics, industrial organisation, and environmental economics. Fourth, we critically assess what is well established and what remains speculative, and we propose a concrete research agenda. The review is written for economists who need the mathematics, for applied mathematicians who want to see their analysis used, and for computational scientists who require reliable solvers. Section 2 sets out the review methodology. Sections 3 to 8 examine the five thematic streams and the fractional frontier. Section 9 reviews numerical methods. Sections 10 and 11 present the critical analysis and future directions, and Section 12 concludes.

REVIEW METHODOLOGY

This review follows a structured, reproducible protocol rather than a narrative selection. We searched Crossref, OpenAlex, Semantic Scholar, and arXiv for records published up to 2026, using combinations of the core terms "partial differential equation", "Hamilton-Jacobi-Bellman", "Fokker-Planck", "Kolmogorov", "Black-Scholes", "mean-field game", "heterogeneous agent", "wealth distribution", "economic geography", "reaction-diffusion", "fractional", and "deep learning". Keyword search was supplemented by backward and forward citation chasing from the foundational contributions [1,8-10].

We applied four inclusion criteria. A study had to be peer-reviewed or a recognised monograph; it had to carry a resolvable digital object identifier (DOI); it had to present mathematical, numerical, or economic content relevant to continuous-time modelling; and it had to be available in English. We excluded editorials, grants, and purely descriptive reports without a model. Every retained record was resolved against Crossref or DataCite, and each registered title was compared with the claimed title; the 100 sources cited here all passed this check and appear with their DOIs in the reference list.

We classified the retained literature into five streams that shape Sections 3 to 8: optimal control and HJB equations; Kolmogorov and Fokker-Planck equations for distributions; PDEs in financial mathematics; mean-field games and mean-field control; and spatial, environmental, and fractional models. Numerical and learning-based solution methods cut across all five and are treated separately in Section 9. We report counts only for the cited corpus, because a formal meta-analysis is inappropriate for heterogeneous modelling studies. The resulting corpus is deliberately broad. Of the 100 unique sources cited here, 23 appeared in 2024 or later and 11 in 2026, so the review reflects the current research frontier rather than only its history; 29 predate 2000 and anchor the mathematical foundations. Roughly one third of the corpus appears in mathematics and applied-mathematics journals, one third in economics and finance journals, and one third in physics, computational, and interdisciplinary venues, which mirrors the genuinely cross-disciplinary character of the field. We used this distribution to guard against two common biases: a purely mathematical reading that ignores economic mechanisms, and a purely economic reading that ignores numerical feasibility. The principal limitation is publication bias toward successful models; failed calibrations and negative results are rarely reported, a point we return to in the critical analysis.

Equation	State variables	Economic role	Representative refs
Hamilton-Jacobi-Bellman	value function $V(x,t)$	dynamic programming, optimal control	[1,5,6,11]
Kolmogorov forward / Fokker-Planck	density $m(x,t)$	evolution of a cross-section of agents	[10,12,13]
Black-Scholes-Merton	option value $V(S,t)$	no-arbitrage pricing of derivatives	[8,14]
Coupled HJB-FP	$V(x,t)$ and $m(x,t)$	mean-field games and mean-field control	[9,15,16]
Reaction-diffusion	activity $u(x,t)$	spatial spread, agglomeration, adoption	[17-19]
Fractional diffusion	density or value	memory, persistence, anomalous transport	[20-22]

Table 1: Canonical PDEs in economics and their roles.

OPTIMAL CONTROL AND HAMILTON-JACOBI-BELLMAN EQUATIONS

From dynamic programming to a nonlinear PDE

The HJB equation is the continuous-time form of Bellman's principle of optimality. For a controlled diffusion $dX = b(X,u)dt + \sigma(X,u)dW$ with running payoff f , the value function V satisfies

$$\rho V = \max_u \{ f(x,u) + \nabla V \cdot b(x,u) + (1/2) \text{tr}(\sigma \sigma^T \nabla^2 V) \},$$

where ρ is the discount rate and u is the control. For deterministic problems the same construction yields the Hamilton–Jacobi equation of the calculus of variations. In economics the equation appears wherever agents allocate resources over time: consumption and saving [1,3,4], portfolio choice [5,23], resource extraction [2], and irreversible investment [24].

The dynamic programming principle states that the value of an optimal policy is unchanged if the agent re-optimises at any intermediate date. Iterating this principle over short intervals and passing to the continuous-time limit yields the HJB equation. The derivation is heuristic because it assumes that V is smooth, and much of the modern theory of stochastic control is concerned with repairing that assumption; the viscosity formulation replaces classical derivatives by test functions and restores existence and uniqueness [6,11]. For economists, the practical consequence is that policy functions may be discontinuous, so comparative statics that assume differentiability must be justified case by case.

The mathematical difficulty is that the value function is generally not differentiable, so the equation must be understood in the viscosity sense [6,7]. This theory, extended to infinite-dimensional and path-dependent settings, provides the existence and uniqueness results on which economic applications rely [25]. Comprehensive treatments of optimal control and viscosity solutions are available in monograph form [11,26]. Life-cycle variants of the Merton problem with borrowing constraints admit near-optimal explicit portfolio rules, which are valuable because they make the HJB solution interpretable.

Economic growth and the maximum principle

Optimal growth remains the clearest economic setting for HJB and Pontryagin methods. The Pontryagin maximum principle and its infinite-horizon variants give necessary conditions that complement the HJB approach and are often more convenient when the state space is unbounded [27]. Historical and modern analyses of the Ramsey–Cass–Koopmans nexus show how the two approaches connect. A subtle but important line of work examines when the value function actually solves the HJB equation, and demonstrates that uniqueness can fail or that the equation can hold only in a weak sense under non-smooth policy rules [28,29]. These results are a warning against treating the HJB equation as automatically well behaved.

Extensions of the HJB framework to richer preferences and constraints are active. Recursive utility, which separates risk aversion from intertemporal substitution, can be written as a continuous-time limit and analysed with stochastic maximum principles. Under borrowing constraints and income adjustment, optimal consumption and investment require state-dependent HJB equations. Delays in the state and control, which are natural in advertising and investment, can be handled with viscosity solutions. Optimal control has also been applied to environmental and biological management, including invasive species and climate policy [30]. Random impulses, which represent abrupt policy interventions or shocks, lead to a generalised HJB equation whose viscosity solution can be characterised when the impulse times are exogenous.

PDES IN FINANCIAL MATHEMATICS

The Black–Scholes–Merton equation and its structure

The Black–Scholes–Merton equation is the archetype of a parabolic PDE in economics. Under geometric Brownian motion and continuous hedging, the arbitrage-free value $V(S,t)$ of a European contingent claim satisfies

$$V_t + (1/2) \sigma^2 S^2 V_{SS} + r S V_S - r V = 0,$$

with terminal payoff at maturity [8,14]. The equation is remarkable because the drift of the underlying asset disappears, so the price depends only on volatility and the riskless rate. The binomial construction gives an intuitive discrete counterpart [31], and the free-boundary version for American options is a parabolic variational inequality [32]. The linear equation admits closed-form solutions, but realistic markets require nonlinear and non-local extensions. American options introduce a free boundary: the holder chooses an exercise time, and the continuation region is separated from the exercise region by an unknown curve. The pricing problem then becomes a parabolic variational inequality, and the boundary satisfies smooth-pasting conditions. Least-squares Monte Carlo provides a simulation-based alternative when the state space is high dimensional and a grid is impractical [32]. This free-boundary structure reappears throughout real-options and investment theory, where the timing of irreversible decisions is the central question [24].

Stochastic volatility, jumps, and nonlinear PDEs

Empirical return distributions are heavy-tailed and volatility is time varying, which breaks the constant-coefficient assumption. Stochastic-volatility models such as Heston's produce a two-dimensional PDE that can be solved analytically for European options [33], while SABR and local-volatility formulations calibrate to implied surfaces [34,35]. Jump-diffusion models add a non-local integral term, turning the PDE into a partial integro-differential equation [36,37], and econometric evidence supports both price and volatility jumps [38]. Extensions allow for fuzzy or mixed-exponential jump laws, which acknowledge that the jump distribution is

itself uncertain and help hedge against model misspecification. Regime switching, stochastic interest rates, and liquidity risk generate coupled systems that are solved with specialised schemes [39,40].

Transaction costs destroy the exact replication argument and introduce a nonlinear, often fully nonlinear, PDE. Early work derived a modified Black–Scholes equation with a nonlinear volatility term [41], and the rigorous treatment of the resulting equation belongs to the theory of viscosity solutions [42]. Related quasi-variational inequalities arise in utility maximisation with optimal stopping under regime switching. These nonlinearities matter because they change both the price and the hedging policy.

Term structure and interest-rate PDEs

Interest-rate modelling is another mature PDE domain. The Vasicek and Cox–Ingersoll–Ross models describe the short rate as a diffusion and yield bond prices as solutions of a parabolic PDE [43,44]. The Heath–Jarrow–Morton framework reformulates the problem in terms of the entire forward curve, producing a stochastic PDE whose drift is fixed by no arbitrage [45]. These models underpin risk management and the valuation of interest-rate derivatives.

DISTRIBUTIONAL DYNAMICS AND CONTINUOUS-TIME MACROECONOMICS

Kolmogorov and Fokker–Planck equations

The second canonical PDE describes not a single agent but an entire distribution. If agents follow $dX = b(X)dt + \sigma(X)dW$, the density $m(x,t)$ evolves by the Kolmogorov forward or Fokker–Planck equation

Subscripts $\partial_t, \partial^2_{xx}$

$$\partial_t m = - \nabla \cdot (b(x) m) + (1/2) \partial^2_{xx} (\sigma^2(x) m),$$

which conserves total mass. In economics this equation governs the evolution of income, wealth, and firm-size distributions, and it is the natural counterpart of the HJB equation for individual behaviour.

Two properties of the forward equation are economically important. First, it conserves total mass, so it is the correct tool when no agent enters or leaves the population; birth, death, and entry require source terms. Second, under mild mixing conditions the density converges to a stationary distribution, which is precisely the object that stationary equilibrium concepts in macroeconomics seek to compute [10]. The stationary Fokker–Planck equation is therefore often solved jointly with a stationary HJB equation, and the pair is the continuous-time analogue of a stationary recursive equilibrium.

Wealth and income distributions

The Fokker–Planck description of wealth has become a central tool. Bouchaud and Mézard showed that multiplicative random growth with exchange generates wealth condensation and power-law tails [46], and kinetic models of money reproduce the exponential bulk of the distribution [47,48]. A sequence of studies established the mathematical foundations of these kinetic equations, including the conditions under which Pareto and Boltzmann–Gibbs laws emerge [12,49,50]. Reviews of econophysics connect the models to empirical income and wealth data, while careful statistical work distinguishes power-law from lognormal behaviour. Firm-size and city-size regularities follow closely related laws [51].

The modern wave of continuous-time macroeconomics embeds these distributional equations inside general equilibrium. The Bewley–Aiyagari–Huggett model, originally stated in discrete time [52,53], can be recast as a coupled HJB–FP system and solved accurately with finite-difference methods [10]. This approach yields analytic characterisations of the consumption of the poor and of the stationary wealth distribution, and has become a workhorse of heterogeneous-agent macroeconomics [54]. Related Fokker–Planck models describe search and matching, liquidity accumulation, and the dynamics of inequality [13,55]. Recent contributions extend the framework to capital-income jumps [56], knowledge and wealth dynamics across markets [57], and first-principles kinetic theories of socio-economic structure [58].

Continuous-time contracting and macrofinance

Continuous-time methods have also transformed contract theory and macrofinance. The principal–agent problem can be solved with a stochastic control formulation in which the agent's continuation value is the state variable [59], building on the linear-contracting results of Holmström and Milgrom. Applied to international finance, these tools clarify how credit flows transmit pecuniary externalities. The common thread is that incentive and equilibrium conditions become drift and boundary conditions in a system of PDEs.

MEAN-FIELD GAMES AND MEAN-FIELD CONTROL

Mean-field games describe strategic interaction among a continuum of symmetric agents. Each agent optimises against the population distribution, and consistency requires that the distribution generated by the optimal behaviour equals the distribution that the agent takes as given. The equilibrium is therefore a coupled forward–backward system: a backward HJB equation for the value function and

a forward Fokker-Planck equation for the density.

Subscripts ∂_t

$$-\partial_t V + H(x, \nabla V, m) = 0, \quad \partial_t m - \nabla \cdot (m \nabla_p H) = 0.$$

The theory was introduced independently by Huang, Malhamé, and Caines and by Lasry and Lions [9,15]. Foundational contributions established well-posedness, long-time behaviour, and the convergence of finite-player games to the mean-field limit [60,61]. The "master equation", a PDE on the space of measures, provides a complete characterisation of the limit and is treated in book-length form [16]. Probabilistic formulations based on forward-backward stochastic differential equations complement the analytic view [62]. Surveys and lecture notes give accessible entry points [63,64]. Existence and uniqueness of mean-field equilibria rest on monotonicity conditions on the coupling between the value function and the density, which play the role that convexity plays in classical optimisation [9,61]. When that coupling is not monotone, multiple equilibria can coexist, and the economic interpretation is that self-fulfilling expectations select among them. The master equation lifts the problem to the space of probability measures and characterises the limit of finite-player games, including the rate of convergence as the number of players grows [16]. This is the mathematically deepest part of the theory and the part least explored in economic applications.

Two features make MFGs especially relevant to economics. First, the mean field is exactly the distributional object that macroeconomists care about, so MFG methods connect naturally to heterogeneous-agent models; mean-field models of segregation and of urban traffic are direct economic applications [65]. Second, the same mathematical structure describes cooperative problems, called mean-field control, which arise in portfolio liquidation and resource management [66]. Recent theory addresses major players, common noise, time inconsistency, portfolio games, and fractional diffusion with anomalous transport [67].

Numerical solution of the coupled system is a field in itself. Finite-difference, semi-Lagrangian, and variational schemes are now standard [68-70], and applications in financial engineering and economic theory are reviewed elsewhere [71]. Reinforcement-learning algorithms can solve mean-field control games in high dimension.

SPATIAL, DIFFUSION, AND ENVIRONMENTAL PDES

Reaction-diffusion and economic geography

Spatial economics uses reaction-diffusion and transport equations to describe the movement of economic activity. The mathematical seeds are Turing's morphogenesis model and Fisher's wave

of advance [17,18], and Beckmann's continuous model of transportation gave an early economic formulation. The new economic geography of Krugman and collaborators recast agglomeration as the outcome of increasing returns and trade costs, and its dynamic extensions are naturally expressed as systems of differential equations [19,72]. Core-periphery dynamics, multi-regional and continuous-space formulations, and the spatial topology of agglomeration form an active subliterate. A reaction-diffusion economic-demographic model predicts spatial pattern formation on geographical scales [73], and stochastic versions exhibit Turing instability [74].

Technology diffusion and optimal transport

Technology adoption and innovation spread are modelled with logistic and Fisher-KPP equations, whose travelling-wave solutions quantify the speed of diffusion [75]. These models connect to the broader reaction-diffusion literature and provide tractable predictions for policy. A parallel strand replaces diffusion with optimal transport: the Benamou-Brenier formulation and Monge-Kantorovich theory provide a geometric framework for matching, sorting, and urban planning [76,77]. Optimal transport also supplies a mean-field interpretation of competitive equilibrium.

Climate and environmental economics

Integrated assessment models combine economic growth with climate dynamics, and their optimal-control formulations are large HJB or Hamilton-Jacobi systems. The social cost of carbon is the shadow price of the climate state [78], and optimal climate policy can be characterised through feedback controls and simple rules [30]. Recent work improves the representation of climate impacts within integrated assessment, which is essential for credible policy analysis [79].

FRACTIONAL AND NONLOCAL PDES IN ECONOMICS

Economic time series often display long memory, and transport in spatial or network settings is frequently anomalous, so the classical integer-order PDE is not always adequate. Fractional derivatives provide a compact description of memory and long-range dependence. The Caputo derivative is the standard operator for initial-value problems, because it accommodates classical initial conditions, while the conformable derivative offers a simpler alternative that preserves the product and chain rules [20,80]. Reviews document the rapid growth of Caputo-type equations and their numerical treatment [81].

Fractional models have been applied across finance and economics. Time- and space-fractional Black–Scholes equations capture heavy tails and memory, and they admit a rich family of numerical schemes [82,83]. Two-sided and Riesz space-fractional diffusion require specialised finite-difference and Galerkin discretisations [21]. A preconditioned policy–Krylov method has been developed for fractional partial integro-differential HJB equations in finance [22], and space-fractional advection–diffusion models have been solved with weighted-residual methods [84]. Analytical time-dependent solutions of transport models that include diffusion, convection, drift, and finite relaxation illustrate how memory effects can be handled in closed form [85]. Fractional mean-field games extend the coupled HJB–FP system to nonlocal diffusion and are beginning to be used in economic and social modelling [67].

The fractional frontier is promising but unsettled. The physical origin of a fractional exponent is often unclear, several operators can fit the same data, and the conformable derivative, although convenient, does not reproduce all classical properties [20,80,81]. Connecting measured exponents to microscopic mechanisms remains an open problem.

NUMERICAL METHODS FOR ECONOMIC PDES

Classical discretisations

Economic PDEs are rarely solvable in closed form, so numerical methods are central. Finite-difference schemes dominate finance and macroeconomics because they are simple, flexible, and easy to combine with free-boundary or variational-inequality formulations [32,86]. Finite-element and finite-volume methods handle irregular domains and are used for multi-asset American options and HJB equations with borrowing fees [87,88]. Spectral and pseudo-spectral methods achieve high accuracy on smooth problems and are attractive for regime-switching and multi-dimensional pricing [89]. When the PDE becomes non-local, as in jump-diffusion or fractional models, these schemes must be adapted, and operator-splitting methods are often used to separate stiff linear and nonlinear parts [100].

The curse of dimensionality

The central obstacle is the curse of dimensionality: a standard grid requires exponentially many points as the number of state variables grows. Three strategies mitigate it. Sparse grids reduce the number of nodes while retaining good accuracy [90]. Model reduction and specialised preconditioners lower the cost of the linear algebra. And when the problem admits a low-dimensional representation, tensor-train or spectral methods compress the solution. These approaches are essential for the high-dimensional HJB equations that arise in portfolio choice, trade execution, and dynamic contracting.

Scientific machine learning

Learning-based solvers have transformed the field over the past decade. Physics-informed neural networks (PINNs) approximate the solution by minimising the PDE residual together with boundary and initial conditions, and they solve both forward and inverse problems without a mesh [91]. The deep backward stochastic differential equation method and its convergence theory provide a probabilistic route to high-dimensional parabolic PDEs [92–94], while the deep Galerkin method learns solutions through a residual objective [95]. Adaptive schemes target high-dimensional HJB equations directly [96], and PINNs have been applied to American option pricing and partial integro-differential equations in finance [97]. Recent work extends PINNs to time-fractional Black–Scholes models with rough volatility [98] and to tempered space-fractional diffusion with exercise-boundary analysis. Reinforcement learning offers a complementary route when the model is unknown or when the control is high dimensional.

Learning-based solvers are attractive for inverse problems, parametric studies, and real-time decision making. Their weaknesses are equally clear: error control is weaker than for classical schemes, training can be costly, and constraints such as mass conservation or the maximum principle are not automatically respected. Hybrid designs that embed a neural network inside a structure-preserving solver, or that use the network only as a surrogate or preconditioner, are the most promising compromise.

Two further ingredients deserve mention. Monte Carlo methods remain indispensable for path-dependent payoffs and for validating PDE solutions, and hybrid schemes that couple a coarse PDE solve with a Monte Carlo correction combine the strengths of both [32,93]. In macroeconomics, simulation of the Fokker–Planck equation provides a transparent check on finite-difference solutions of heterogeneous-agent models, and the two approaches are now often used together [10].

Method	Typical dimension	Strengths	Limitations	Representative refs
Finite differences	Low	Simple, free boundaries, stability theory	Grid cost grows rapidly	[32,86]
Finite elements	Low–medium	Complex geometry, error analysis	Meshing, stiffness	[87,88]
Spectral methods	Low–medium	High accuracy on smooth data	Sensitive to discontinuities	[89]
Sparse grids	Medium–high	Breaks the curse of dimensionality	Complexity of construction	[90]

Operator splitting	Any	Handles non-local and stiff terms	Splitting error	[98]
Deep BSDE / DGM	High	Mesh-free, high dimension	Weak error control	[92-95]
PINNs	High	Inverse problems, data fusion	Training cost, constraint violation	[91,97,98]

Table 2; Representative numerical strategies for economic PDEs.

DISCUSSION

This section interprets the evidence assembled in Sections 3 to 9. We first present a critical analysis of the modelling literature, then draw out the implications for research practice and policy, and finally suggest future research directions.

Critical analysis of the literature

A century of work has established PDEs as a rigorous language for economic dynamics, and the last two decades have extended them to populations, space, and data-driven settings. A critical reading of the literature nevertheless reveals several persistent weaknesses.

The first is validation. Economic PDE models are frequently assessed qualitatively, by showing that simulated distributions or price surfaces "look reasonable", rather than by quantitative out-of-sample tests. Calibrated continuous-time macro models are an exception, but even there parameter uncertainty is seldom propagated into policy conclusions [10]. The community should adopt standard benchmarks and report error metrics against held-out data.

The second is the curse of dimensionality. Most economic problems have many state variables, and classical grids fail well before the dimension of practical interest. Learning-based solvers relieve but do not eliminate the problem, because their error control remains weak and their training cost grows with dimension and required accuracy [91,93,97]. Progress requires solvers whose complexity is polynomial in dimension for structured problems, and clear statements of the structure that makes this possible.

The third is identification. Fokker-Planck and mean-field models often contain several parameters that trade off against one another, so the same distributional moments can be reproduced by different combinations. Fractional models amplify this problem because the order of the operator and the diffusion coefficient are strongly correlated [21,22]. Formal identifiability analysis and Bayesian calibration should become routine.

The fourth is the treatment of boundaries and constraints. Economic state spaces have natural boundaries: wealth cannot be negative, consumption cannot exceed resources, and probabilities must remain non-negative. Many numerical schemes violate these bounds, producing spurious behaviour. Structure-preserving discretisations, including maximum-principle-preserving and positivity-preserving schemes, are well developed for physical PDEs but are only beginning to be standard in economics.

The fifth is uncertainty quantification. Policy advice based on a single calibrated trajectory overstates confidence. The stochastic and mean-field literature provides the machinery for distributional analysis [61], but it is rarely used to report predictive intervals. Bridging this gap is a priority.

Finally, reproducibility remains uneven. Some models are implemented in open, well-documented code, but many published results depend on undocumented parameters or proprietary data. Shared benchmarks and standard input formats would accelerate cumulative progress.

Synthesis and implications

Taken together, the literature shows a field that is mathematically mature but unevenly validated. Its analytic core, viscosity solutions for HJB equations, well-posedness for Kolmogorov-Fokker-Planck equations, and no-arbitrage for parabolic pricing equations, is settled and reliable. The applied frontier is less secure. Economic PDE models are often calibrated once and then trusted across regimes, and their dimensionality, nonlinearity, and non-locality strain both theory and computation. For practitioners, the practical implication is that model outputs should carry uncertainty bands and be stress-tested against alternative specifications rather than presented as point forecasts. For theorists, existence and uniqueness results matter only when they are paired with schemes that preserve the economic structure of the problem. The most valuable contributions of the next decade are therefore likely to sit at the interface of these two communities, combining rigorous analysis with structure-preserving computation and transparent validation.

Future Research Directions

Several directions follow from the analysis above and from trends visible in 2024–2026 publications.

Hybrid physics–learning solvers should be developed and analysed. The most promising designs embed conservation laws, the maximum principle, or no-arbitrage identities directly into the learning objective, or use classical solvers as teachers. This combines the flexibility of neural networks with the reliability of structure-preserving schemes [91,94].

Structure-preserving high-dimensional schemes deserve attention. Positivity-preserving and mass-conserving discretisations for coupled HJB–FP systems, including fractional and nonlocal variants, would remove a major source of numerical error and would make long-horizon simulations trustworthy [22,67,69].

Fractional and nonlocal economics needs a firmer empirical foundation. The link between measured exponents, memory, and microscopic mechanisms should be established with experiments and identification strategies rather than assumed [20,21,81]. Distributed-order and tempered operators offer flexible descriptions that may fit data with fewer ad hoc assumptions.

Mean-field methods should be integrated more deeply with macroeconomics and finance. The master equation provides a general equilibrium theory of populations [16], and its numerical approximation is now feasible [70]. Applications to climate transition, digital markets, and systemic risk are natural next steps [30,78].

Optimal transport and spatial PDEs offer a unifying geometric language for sorting, matching, and the movement of goods and people [76,77]. Coupling these models with reaction–diffusion dynamics and with data on mobility could yield testable theories of agglomeration and regional divergence [74].

Inverse problems and data fusion form a fourth direction. Many economic parameters, risk premia, and transition intensities are only weakly identified by prices or aggregate quantities, yet PDE models can be combined with rich micro data through inverse-problem formulations. Physics-informed methods are well suited to this task because they incorporate observations alongside the governing equation [91,97]. Embedding economic restrictions such as no arbitrage or budget balance directly into the loss removes otherwise common spurious solutions, and it makes the estimated parameters economically interpretable.

Finally, the field would benefit from open benchmarks. A curated set of economic PDE problems, the Ramsey–Cass–Koopmans model, the Bewley–Aiyagari–Huggett equilibrium, a Heston calibration, a mean-field game of segregation, and a fractional Black–Scholes pricing problem, each with reference parameters and expected observables, would let competing methods be compared fairly. Standardisation of this kind has transformed other computational fields and would accelerate progress here.

CONCLUSION

Partial differential equations provide a rigorous and flexible foundation for economic dynamics. The HJB equation formalises intertemporal optimisation, the Kolmogorov and Fokker–Planck equations describe the evolution of populations, the Black–Scholes–Merton equation and its nonlinear extensions price risk, and coupled HJB–FP systems capture strategic interaction among large populations. Reaction–diffusion and transport equations organise spatial and environmental economics, while fractional operators encode memory and anomalous transport. The literature, represented here by 100 verified sources, is mathematically deep and economically consequential.

Our critical assessment finds that the field is mathematically mature but empirically under-validated, with unresolved challenges in dimensionality, parameter identification, constraint preservation, and uncertainty quantification. The most promising responses combine the reliability of structure-preserving numerical schemes with the flexibility of scientific machine learning, embed economic structure such as conservation and no arbitrage into solvers, and adopt shared benchmarks that make results comparable. If these directions are pursued, PDE-based economic modelling can move from elegant illustration to quantitative prediction, which is the standard that modern policy and financial decision making demands.

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AUTHOR CONTRIBUTIONS

The author conceived the review, designed the methodology, performed the literature search and verification, and wrote the manuscript.

CONFLICT OF INTREST

The author declares no competing interests.

DATA AVAILABILITY

All bibliographic records, verification logs, and the compiled manuscript are available in the accompanying workspace. Reference metadata were resolved through Crossref and DataCite.

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