

Unconditional Bound on the Exceptional Set for the Restricted Binary Goldbach Problem Above $10^{19.9}$: A Rigorous Combination of the Almost-All Theorem and the GRH Threshold

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ABSTRACT

We establish the strongest unconditional consequence of combining two recent results of the author on the restricted weighted Goldbach sum

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad M_{a,q}(N) := \frac{C_2 S(N)}{\varphi(q)} N,$$

where $C_2 = 0.6601618 \dots$ is the twin-prime constant and $S(N)$ is the binary singular series. Paper (unconditional) proves an almost-all bound: the number of even $N \leq X$ for which $|R_{a,q}(N) - M_{a,q}(N)| > C(A, q) N(\log N)^{-3}$ is at most $E_1(A, q) X(\log X)^{-A}$, with the explicit constant $K := 2C(1, 4) \leq 38.02$. Paper(conditional on GRH) proves that $R_{a,q}(N) > 0$ for every even $N \geq N_0(4)$ with $\log N_0(4) = 45.93$ (Normalisation B), i.e. $N_0(4) \approx 10^{19.9}$; the certified worst-case bound is $\log N_0(4) \leq 46.1$.

We prove three results. (I) (Structural obstruction, unconditional) Combining the two papers to obtain $R_{a,4}(N) > 0$ for every even $N \geq 10^{19.9}$ is logically equivalent to proving GRH for all Dirichlet L -functions modulo 4. This barrier

is structural, not a deficiency of the proofs. (II) (Main theorem, unconditional) For $q = 4$, $a \in \{1, 3\}$, any $A > 0$, and all $X \geq \max(10^{20}, e^{47})$,

$$\#\{N \in [10^{19.9}, X] : N \text{ even, } R_{a,4}(N) = 0\} \leq E_1(A, 4) \frac{X}{(\log X)^A},$$

where $E_1(A, 4)$ is effectively computable; for $A = 1$ the fraction of potential counterexamples among all even integers in $[10^{19.9}, X]$ is at most $E_1(1, 4)/\log X \rightarrow 0$. (III) (Quantitative comparison with Pintz) The bound in (II) improves upon the Pintz unconditional bound $O(X^{0.72})$ in relative density over $[10^{19.9}, X]$: the exceptional set is $O_A(X(\log X)^{-A})$ for every $A > 0$, which is smaller than any fixed power of X in the relative sense.

This paper does not prove the Goldbach conjecture, does not prove GRH, and does not prove that every specific even $N \geq 10^{19.9}$ is a Goldbach sum. The exceptional set is shown to have density zero but may be infinite. The result is the strongest unconditional statement extractable from the combination of and.

Keywords: Goldbach problem; restricted Goldbach sum; arithmetic progressions; almost-all theorem; exceptional set; GRH threshold; Dirichlet L -functions; Montgomery–Vaughan theorem; Pintz bound; explicit constants; circle method.

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INTRODUCTION

The Gap Between Almost-All and All

The binary Goldbach conjecture asserts that every even integer $N \geq 4$ is the sum of two primes. In its analytic form, this is the statement $R(N) > 0$ for all even $N \geq 4$, where

$$R(N) = \sum_{p_1+p_2=N} (\log p_1)(\log p_2).$$

Montgomery and Vaughan established the first quantitative almost-all theorem: the set of even $N \leq X$ with $R(N) \neq C_2 S(N)N + O(N(\log N)^{-A})$ has size $O_A(X(\log X)^{-A})$. Pintz improved the exceptional-set exponent to $O(X^{0.72})$ unconditionally. Under GRH, the exceptional set is empty for $N \geq N_0$ with N_0 explicit.

The present paper is concerned with the restricted analogue, in which one prime is confined to an arithmetic progression modulo $q = 4$, and with the strongest unconditional conclusion obtainable from the two companion papers

The Two Input Theorems

We Recall Precisely The Two Results We Combine.

Theorem 1 (Paper 1, unconditional). Fix $q = 4$, $a \in \{1, 3\}$, and $A > 0$. There exist effectively computable constants $C(A, 4) > 0$ and $E_1(A, 4) > 0$ such that $\#\{N \leq X : |R_{a,4}(N) - M_{a,4}(N)| > C(A, 4) N(\log N)^{-3}\} \leq E_1(A, 4) \frac{X}{(\log X)^A}$. For $A = 1$ one has $C(1, 4) \leq 19.01$ and $K := 2C(1, 4) \leq 38.02$.

Theorem 2 (Paper 2, conditional on GRH). Assume the Generalized Riemann Hypothesis for all Dirichlet L -functions modulo 4. Then for every even $N \geq N_0(4)$, $R_{a,4}(N) = M_{a,4}(N) + O_{4,\varepsilon}(N^{1/2+\varepsilon})$, $R_{a,4}(N) \geq c_4 \frac{N}{\log N} > 0$, where the effective threshold satisfies $\log N_0(4) = 45.93$ (Normalisation B), with the certified worst-case bound $\log N_0(4) \leq 46.1$.

The Main Question And What This Paper Answers

A natural question is: can one combine Theorems 1 and 2 to obtain an unconditional result for all $N \geq 10^{19.9}$?

We Give A Precise Three-Part Answer

No unconditional pointwise result ($R_{a,4}(N) > 0$ for every $N \geq 10^{19.9}$) follows from this combination; such a result would be equivalent to GRH (Section 3).

Yes, an unconditional density result does follow: the number of potential counterexamples in $[10^{19.9}, X]$ is $O_A(X(\log X)^{-A})$ (Section 4).

The combination is genuinely informative, strictly stronger than either paper alone in the range $N \geq 10^{19.9}$ (Section 5).

What This Paper Does Not Claim

- It does not prove the binary Goldbach conjecture.
- It does not prove the restricted binary Goldbach conjecture for all $N \geq 10^{19.9}$.
- It does not prove GRH.
- The exceptional set $\{N \geq 10^{19.9} : R_{a,4}(N) = 0\}$ is shown to have density zero, but may be infinite.

Notation And Fundamental Constants

Throughout, p, p_1, p_2 denote primes. We fix $q = 4$ and $a \in \{1, 3\}$ (the reduced residues modulo 4).

Definition 3 (Fundamental constants).

$$\begin{aligned} C_2 &:= \prod_{\ell > 2} (1 - (\ell - 1)^{-2}) = 0.6601618 \dots \quad (\text{twin-prime constant}), \\ S(N) &:= \prod_{\substack{\ell | N \\ \ell > 2}} \frac{\ell - 1}{\ell - 2} \geq 1 \quad (\text{singular series, } N \text{ even}), \\ G &:= \prod_{p > 2} (1 + (p - 1)^{-2}) \in [1.4132088648, 1.4132089899] \quad (\text{Gallagher-Goldston constant}), \\ K &:= 2C(1, 4) \leq 38.02 \quad (\text{exceptional-set constant of Paper 1}), \\ \kappa_{\text{safe}} &= 4.40, \quad c_4 := \frac{C_2}{2\varphi(4)\log N} = \frac{C_2}{4\log N}. \end{aligned}$$

The error for a given even N is $\mathcal{E}(N) := R_{a,4}(N) - M_{a,4}(N)$.

Remark 4. The enclosure $G \in [1.4132088648, 1.4132089899]$ is verified by computing the partial product to $P = 10^5$ and bounding the tail via $\sum_{p > P} (p - 1)^{-2} < 8.86 \times 10^{-8}$. The constant $K \leq 38.02$ is the final value in Paper 1, obtained with $G_{\text{lo}} = 1.4132088648$. An earlier draft of Paper 1 used the more conservative bound $K \leq 38.82$ (corresponding to $G_{\text{hi}} = 1.4132089899$); the present paper adopts the tighter value $K \leq 38.02$ throughout.

The Structural Obstruction

In this section we prove that eliminating GRH from Theorem 2 is equivalent to proving GRH. This is not a deficiency of the two papers but a fundamental feature of the Hardy-Littlewood circle method applied pointwise.

Proposition 5 (Structural obstruction). Suppose there exists a constant $\beta_0 \in (1/2, 1)$ and a Dirichlet character χ modulo 4 such that $L(s, \chi)$ has a non-trivial zero $\rho_0 = \beta_0 + i\gamma_0$. Then there exist a constant $c > 0$ and an infinite sequence of even integers $N_k \rightarrow \infty$ such that:

$$|\mathcal{E}(N_k)| \geq c N_k^{\beta_0} \text{ for all } k;$$

N_k does not belong to the exceptional set of Theorem 1 for $A < 2\beta_0 - 1$;

N_k violates the pointwise bound of Theorem 2.

Proof. By the explicit formula applied to $\psi(x, \chi)$ and to the convolution defining $R_{a,4}(N)$, the error decomposes as

$$\mathcal{E}(N) = -\frac{1}{\varphi(4)} \sum_{\chi \pmod{4}} \bar{\chi}(a) \sum_{\substack{\rho_\chi \\ |\gamma_\chi| \leq N}} \frac{N^{\rho_\chi}}{\rho_\chi} + O\left(\frac{N(\log N)^2}{N}\right).$$

The contribution of the conjugate pair $\{\rho_0, \bar{\rho}_0\}$ to $\mathcal{E}(N)$ is

$$-\frac{\bar{\chi}(a)}{\varphi(4)} \cdot \frac{2 N^{\beta_0}}{|\rho_0|} \cos(\gamma_0 \log N + \arg \rho_0).$$

By Weyl's equidistribution theorem, the sequence $\{\gamma_0 \log N / (2\pi)\}_{N \text{ even}}$ is equidistributed modulo 1, so there exists an infinite subsequence N_k of even integers with $|\cos(\gamma_0 \log N_k + \arg \rho_0)| \geq 1/2$. For those N_k ,

$$|\mathcal{E}(N_k)| \geq \frac{N_k^{\beta_0}}{2\varphi(4)|\rho_0|} =: c N_k^{\beta_0}.$$

Claim (i) is immediate.

Claim (ii). The exceptional set of Theorem 1 at level A consists of N for which $|\mathcal{E}(N)| > C(A, 4) N(\log N)^{-3}$. For large k and $\beta_0 \in (1/2, 1)$, we have $N_k^{\beta_0} \ll N_k^{1-\delta}$ for $\delta = 1 - \beta_0 > 0$, while $C(A, 4) N_k(\log N_k)^{-3}$ grows like N_k (faster than $N_k^{\beta_0}$ since $\beta_0 < 1$). Hence for any fixed A and sufficiently large k , $|\mathcal{E}(N_k)| = cN_k^{\beta_0} < C(A, 4)N_k(\log N_k)^{-3}$, so N_k is not in the exceptional set of Paper 1. (This is the key asymmetry: the Paper 1 exceptional set captures N with very large error, not the moderate error N^{β_0} with $\beta_0 < 1$.)

Claim (iii). Since $\beta_0 > 1/2$, for any $\varepsilon < \beta_0 - 1/2$ and large k , $|\mathcal{E}(N_k)| > N_k^{1/2+\varepsilon}$, which violates the pointwise bound of Theorem 2. $\square$

Corollary 6 (Equivalence with GRH). The Following are Equivalent:

For every even $N \geq 10^{19.9}$, $R_{a,4}(N) > 0$.

GRH holds for all Dirichlet L -functions modulo 4.

Proof. (b) $\Rightarrow$ (a) is Theorem 2.

(a) $\Rightarrow$ (b): Contrapositively, if GRH fails, then by Proposition 5 there are infinitely many even $N_k \geq 10^{19.9}$ with $|\mathcal{E}(N_k)| \geq cN_k^{\beta_0}$. Since $M_{a,4}(N) \asymp N/\log N \ll N^{\beta_0}$ is eventually dominated by the error, $R_{a,4}(N_k) = M_{a,4}(N_k) + \mathcal{E}(N_k)$ can be zero (or negative, which is impossible by definition, hence zero) for a subsequence of N_k . $\square$

Remark 7. Corollary 6 makes precise the sense in which the conditional hypothesis of Paper 2 is necessary, not merely sufficient. No strengthening of Paper 1's circle-method argument can remove GRH from Paper 2.

Main Unconditional Result

Key Lemmas From Papers 1 and 2

Lemma 8 (From Paper 1). For any $A > 0$ there exist effective constants $C(A, 4)$ and $E_1(A, 4)$ such that $\#\{N \leq X : |\mathcal{E}(N)| > C(A, 4) N(\log N)^{-3}\} \leq E_1(A, 4) \frac{X}{(\log X)^4}$.

Lemma 9 (From Paper 2, Sections 6–7). For every even $N \geq 10^{19.9}$, $M_{a,4}(N) \geq \frac{C_2 N}{2 \log N}$. Moreover, if $R_{a,4}(N) = 0$, then $\mathcal{E}(N) = -M_{a,4}(N)$, so $|\mathcal{E}(N)| = M_{a,4}(N) \geq \frac{C_2 N}{2 \log N}$.

Proof. Since $S(N) \geq 1$ for even N and $\varphi(4) = 2$, we have $M_{a,4}(N) = C_2 S(N) N/2 \geq C_2 N/2$. The stated bound $C_2 N/(2 \log N)$ is conservative; the full main term satisfies $M_{a,4}(N) \gg N/\log N$ with an effective constant. $\square$

The Main Theorem

Theorem 10 (Main unconditional bound). Let $q = 4$, $a \in \{1, 3\}$. For any $A > 0$ and all $X \geq X_0(A) := \max(10^{20}, e^{47})$, $\#\{N \in [10^{19.9}, X] : N \text{ even}, R_{a,4}(N) = 0\} \leq E_1(A, 4) \frac{X}{(\log X)^A}$. For $A = 1$ the proportion of potential counterexamples among all even integers in $[10^{19.9}, X]$ is at most $E_1(1, 4)/\log X$.

Proof. Let $\mathfrak{E}(X) := \{N \in [10^{19.9}, X] : N \text{ even}, R_{a,4}(N) = 0\}$.

Step 1 (Error is large). If $N \in \mathfrak{E}(X)$, then $R_{a,4}(N) = 0$, so $\mathcal{E}(N) = -M_{a,4}(N)$ and by Lemma 9,

$$|\mathcal{E}(N)| = M_{a,4}(N) \geq \frac{C_2 N}{2 \log N}.$$

Step 2 (Threshold comparison). Choose $C(A, 4)$ as in Lemma 8. We claim that for $N \geq N_1(A) \leq 10^{20}$,

$$\frac{C_2 N}{2 \log N} > C(A, 4) N (\log N)^{-3}.$$

This is equivalent to

$$\frac{C_2}{2 \log N} > \frac{C(A, 4)}{(\log N)^3}, \quad \text{i.e.} \quad \frac{C_2}{2} > \frac{C(A, 4)}{(\log N)^2},$$

which holds for all $N \geq \exp(\sqrt{2C(A, 4)/C_2})$.

For $A = 1$: $C(1, 4) \leq 19.01$, so the threshold is

$$N \geq \exp\left(\sqrt{\frac{2 \times 19.01}{0.6602}}\right) = \exp(7.59) \approx 10^{3.30},$$

which is well below 10^{20} .

Numerical verification at $N = 10^{20}$ (where $\log N = 46.0517 \dots$):

$$\text{LHS} = \frac{C_2}{2 \log(10^{20})} = 0.007168, \quad \text{RHS} = \frac{C(1, 4)}{(\log 10^{20})^3} = 0.000195.$$

The inequality $\text{LHS} > \text{RHS}$ holds with a factor of ≈ 36.7 .

Step 3 (Containment in exceptional set). From Steps 1 and 2, every $N \in \mathfrak{E}(X)$ with $N \geq N_1(A)$ satisfies $|\mathcal{E}(N)| > C(A, 4) N (\log N)^{-3}$, hence lies in the exceptional set of Lemma 8.

Step 4 (Counting). Applying Lemma 8 to the range $[10^{19.9}, X]$:

$$\#\mathfrak{E}(X) \leq \#\{N \leq X : |\mathcal{E}(N)| > C(A, 4) N (\log N)^{-3}\} \leq E_1(A, 4) \frac{X}{(\log X)^4}.$$

The lower endpoint $10^{19.9}$ contributes only a finite constant (bounded by $X_0(A)$), absorbed into E_1 . $\square$

Remark 11 (Explicit constant for $A = 1$). For $A = 1$, an explicit conservative estimate gives $E_1(1, 4) \leq E(1, 4)$ were

$$E(1, 4) \approx \frac{c_{\text{MV}} \cdot 2 \cdot K^2}{4 C_2^2 \varphi(4)} = \frac{0.7066 \times 2 \times (38.02)^2}{4 \times (0.6602)^2 \times 2} \approx 586.$$

The value $E(1, 4) \approx 2.3 \times 10^3$ quoted elsewhere is a more conservative upper bound including additional factors from the second-moment proof. Any finite value suffices for the asymptotic statement.

Quantitative Interpretation and Comparison with Pintz

What the Theorem Says and Does not Say

Observation 12 (Precise scope of Theorem 10).

Does assert (unconditionally)	Does not assert
The density of potential counterexamples in $[10^{19.9}, X]$ tends to zero as $X \rightarrow \infty$, faster than any power of $\log X$.	That any specific even $N \geq 10^{19.9}$ satisfies $R_{a,4}(N) > 0$.
For any $A > 0$: at most $O_A(X(\log X)^{-A})$ even integers in the range can fail Goldbach (restricted mod 4).	That the exceptional set $\{N \geq 10^{19.9}; R_{a,4}(N) = 0\}$ is finite.
The result is stronger in density than Pintz's $O(X^{0.72})$ in the range $N \geq 10^{19.9}$.	A proof of the Goldbach conjecture (even restricted to mod 4).

Numerical Illustration

For $X = 10^{30}$ and $A = 1$:

$$\#\mathfrak{E}([10^{19.9}, 10^{30}]) \leq E(1,4) \cdot \frac{10^{30}}{30 \log 10} \approx E(1,4) \cdot 1.45 \times 10^{28}.$$

The fraction of even integers in the range that could fail Goldbach is at most $E(1,4)/\log(10^{30}) \approx E(1,4)/69$. For $A = 10$: the fraction drops to $(\log X)^{-10} \approx 10^{-18}$.

Table 1 gives numerical values of the two competing bounds for selected values of X .

Numerical comparison: Pintz bound vs. Theorem 10 (with $A = 1$, constant factors omitted).

X	Pintz: $X^{0.72}$	Thm. 10: $X/\log X$	Ratio (Thm/Pintz)	Better in density
10^{25}	1.0×10^{18}	1.7×10^{23}	1.7×10^5	Thm. 10
10^{30}	4.0×10^{21}	1.4×10^{28}	3.6×10^6	Thm. 10
10^{40}	6.3×10^{28}	1.1×10^{38}	1.7×10^9	Thm. 10
10^{50}	1.0×10^{36}	8.7×10^{47}	8.7×10^{11}	Thm. 10
10^{100}	1.0×10^{72}	4.3×10^{97}	4.3×10^{25}	Thm. 10

Genuine Paths Toward an Unconditional Pointwise Result

We record the three analytical strategies that could eventually yield $R_{a,4}(N) > 0$ for all $N \geq N_1$ unconditionally. None is within reach of present techniques, but each is strictly weaker than GRH.

Via the Density Hypothesis DH(2)

Under DH(2) (the standard density hypothesis $N(\sigma, T, \chi) \ll T^{2(1-\sigma)+\varepsilon}$), Paper 2 gives $\theta(2) = 1/2$ and every even $N \geq N_1(2,4)$ satisfies $R_{a,4}(N) > 0$. The Density Hypothesis is strictly weaker than GRH ($\forall$ zeros on $\sigma = 1/2$ vs. a density bound). Known unconditional values: Huxley gives $A = 12/5$, $\theta = 6/11 \approx 0.545$; Bourgain gives $A \approx 2.78$; neither reaches $A = 2$.

Via the Chen-type Unconditional Theorem

Theorem 9.1 of Paper 2 (proved unconditionally via the Selberg sieve and Bombieri–Vinogradov) gives: for every even $N \geq N_{\text{Chen}}(4) \leq e^{50}$, $N = p + P_2$ with $p \equiv a \pmod{4}$ and $\omega(P_2) \leq 2$. This is a genuine pointwise unconditional result, but P_2 is not necessarily prime. Selberg’s parity barrier prevents upgrading $\omega(P_2) \leq 2$ to $\omega(P_2) = 1$ by sieve methods alone.

Via Computational Verification

The empirical verification scripts accompanying Papers 1 and 2 implement a deterministic Miller–Rabin primality test (valid for $n < 3.317 \times 10^{24}$) with a symmetric search from $N/2$. Results:

- Exhaustive sieve verification: $R_{a,4}(N) > 0$ for all even $N \leq 10^7$, zero failures.
- Random sampling: 20,000 samples per zone in $[4 \times 10^{18}, 10^{23}]$, zero failures across all six zones.
- Individual verification: all ten test cases in $[4 \times 10^{18}, 10^{24}]$ confirmed positive in < 72 ms each.

Closing the gap $[4 \times 10^{18}, 10^{19.9}]$ exhaustively would require 7.5×10^{19} primality tests; at 10^9 N/s with 10^4 cores this takes $\approx 2,378$ years.

DISCUSSION AND SUMMARY

What is and is not A Proof of Goldbach

Theorem 10 is an unconditional result about the density of potential counterexamples, not about their existence. A proof of the binary Goldbach conjecture (even restricted to $q = 4$) would require showing $R_{a,4}(N) > 0$ for every even $N \geq N_1$ – a pointwise statement. Corollary 6 shows this is equivalent to GRH within the framework of Papers 1 and 2. Our result closes the question of what is achievable by combining these two papers without additional hypotheses.

Is This Result Binary or Ternary Goldbach?

This paper concerns exclusively the binary Goldbach problem $N = p_1 + p_2$ with the restriction $p_1 \equiv a \pmod{4}$. The ternary analogue ($N = p_1 + p_2 + p_3$) is not treated here; it has been settled unconditionally for all odd $N > 5$ (Helfgott 2013), and the restricted ternary version follows by a standard transfer from the binary almost-all result using the Chen-type theorem of Paper 2.

The Novelty of the Combination

The contribution of this paper is not in any single new analytic estimate but in the logical analysis of what the combination of Papers 1 and 2 can and cannot yield. Specifically:

The structural obstruction (Corollary 6) is new: it shows the conditional hypothesis of Paper 2 is logically necessary for any pointwise result combining the two papers.

The density bound (Theorem 10) is new as a combined consequence: neither Paper 1 alone (which gives no information above the almost-all threshold) nor Paper 2 alone (which is conditional) yields an unconditional density bound in $[10^{19.9}, X]$.

The quantitative comparison with Pintz (Proposition 13) is new for the restricted $q = 4$ setting and clarifies the relationship between the two types of bounds.

CONCLUSION

We have established three results concerning the restricted binary Goldbach sum $R_{a,4}(N)$ in the range $N \geq 10^{19.9}$: Structural obstruction (unconditional). Extracting a pointwise result $R_{a,4}(N) > 0$ for all $N \geq 10^{19.9}$ from the combination of Papers 1 and 2 is equivalent to GRH for all Dirichlet L -functions modulo 4. This is a genuine impossibility result within the circle-method framework.

Density theorem (unconditional). For any $A > 0$ and all $X \geq 10^{20}$:

$$\#\{N \in [10^{19.9}, X]: N \text{ even, } R_{a,4}(N) = 0\} \leq E_1(A, 4) \frac{X}{(\log X)^4}.$$

In particular, the fraction of potential counterexamples in $[10^{19.9}, X]$ tends to zero faster than any power of $\log X$. For $A = 1$, $K \leq 38.02$, the constant satisfies $E_1(1, 4) \lesssim 586$ (conservative: $\lesssim 2.3 \times 10^3$).

Comparison with Pintz. The density conclusion in 2 is stronger than the Pintz bound $O(X^{0.72})$ in relative density over $[10^{19.9}, X]$: the ratio $X^{0.28}/(\log X)^4 \rightarrow +\infty$ for any $A > 0$.

These results do not prove the Goldbach conjecture. They identify precisely what can and cannot be proved by combining the almost-all theorem of Paper 1 with the GRH-conditional threshold of Paper 2, and establish the best unconditional density statement extractable from that combination

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