

Restricted Goldbach Sums in Arithmetic Progressions: Analytic Hierarchy, Sub-Exponential Bounds, and Riemann Zero Detection

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ABSTRACT

We develop a unified and fully audited analytic hierarchy for the restricted weighted Goldbach sum

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad q \geq 1, \gcd(a, q) = 1,$$

with expected main term $M_{a,q}(N) := C_2 \mathfrak{S}(N) N/\varphi(q)$, and exceptional set $\mathcal{E}_{a,q}(X) := \{N \leq X, \text{Neven: } R_{a,q}(N) = 0\}$. The paper consolidates and supersedes preprint version 3, integrating results from Papers 1, 9 and 14 of the Anderson Series, with all documented corrections applied.

The unconditional core establishes three nested levels. Level 1 is an effective almost-all theorem via the standard L^4 minor-arc route, with explicit constant $K = 2C(1,4) \leq 38.82$. Level 1.5 is a sub-exponential exceptional-set bound $\#\mathcal{E}_{a,q}(X) \ll_q X \exp(-\sqrt{\log X/R})$ with Stechkin's constant $R = 9.6459$, proved unconditionally by absorbing any potential Siegel zero into a modified main term. Level 1.5+ is a Hölder minor-arc refinement giving the improved constant $K_{\text{new}} \leq 9.80$ and, for moduli $q \leq 200$ certified free of Siegel zeros, an unconditional pointwise sub-exponential bound with $C(4) \leq 120$ and $\log N_0(4) \leq 42$.

Three structural obstructions (Double-Pole, Borel-Cantelli, ETK Dimensional Explosion) formally retract three classical routes to unconditional finiteness. Under DH and GRH, conditional hierarchies (with $\theta(A) = 1 - 2/(A + 2)$ and $\log N_0(4) = 45.93$) are recorded. The Gowers-Spectral Bridge gives conditional finiteness under the Uniform Spectral Gap (USG) hypothesis with effective threshold $N_0(4) \leq 10^{16}$.

Keywords: Goldbach problem; arithmetic progressions; restricted Goldbach sums; almost-all theorem; explicit constants; Hölder minor-arc bound; sub-exponential exceptional set; Siegel-zero absorption; Stechkin zero-free region; circle method; ternary Goldbach; Density Hypothesis; GRH; Gowers-Spectral Bridge; Uniform Spectral Gap; structural obstructions.

PART I: THE UNCONDITIONAL CORE

INTRODUCTION

The restricted binary Goldbach problem

Goldbach's binary conjecture, in weighted analytic form, asserts that every sufficiently large even integer N satisfies

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$$R(N) := \sum_{p_1+p_2=N} (\log p_1)(\log p_2) \sim \mathfrak{S}(N) N, N \rightarrow \infty, \quad (1)$$

where $\mathfrak{S}(N) = 2C_2 \prod_{p|N, p>2} (p-1)/(p-2)$ is the Hardy-Littlewood singular series and

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.6601618 \dots$$

is the twin-prime constant. We study the restricted variant

$$R_{a,q}(N) := \sum_{\substack{p_1+p_2=N \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2), \quad (2)$$

with expected main term

$$M_{a,q}(N) := \frac{C_2 \mathfrak{S}(N)}{\varphi(q)} N. \quad (3)$$

The error and exceptional set are

$$\mathcal{E}_{a,q}(N) := R_{a,q}(N) - M_{a,q}(N), \quad \mathcal{E}_{a,q}(X) := \{N \leq X, \text{ Neven: } R_{a,q}(N) = 0\}.$$

Historical benchmarks

This paper made use of previous work by the author himself [1]. The almost-all theory of the binary Goldbach problem traces back to Van der Corput, Estermann and Chudakov in the 1930s. Hardy and Littlewood [3] (1923) heuristically predicted the asymptotic in (1) and introduced the singular series. Vinogradov [12] gave the definitive circle-method treatment. Lavrik established the first quantitative almost-all theorem with explicit logarithmic saving. Montgomery and Vaughan [6] (1975) proved the power-saving exceptional-set bound $\ll X^{1-\delta}$. Liu, Liu and Wang extended the theory to arithmetic progressions. Pintz [7] refined the unconditional exceptional-set exponent to $X^{0.72}$.

The Anderson Series hierarchy

This paper organises the surviving results of the series as follows:

Unconditional almost-all theorem with via the standard route.

Sub-exponential exceptional-set bound, with Siegel-zero absorption.

Hölder minor-arc refinement, giving, and an unconditional pointwise sub-exponential bound for with.

Conditional results under the Density Hypothesis and GRH.

Unconditional features

Part I uses no Generalised Riemann Hypothesis, no zero-density hypothesis, no ternary sums, and no Chen-type sieve. All numerical constants are certified via partial Euler products with explicit Mertens-type tails.

NOTATION AND CERTIFIED CONSTANTS

Throughout, p, p_1, p_2 denote primes; Λ is the von Mangoldt function; μ is the Möbius function; φ is Euler's totient; $e(\alpha) := e^{2\pi i \alpha}$; $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$. For functions f, g , $f \ll g$ means $|f| \leq Cg$ for an absolute constant $C > 0$; subscripts indicate allowed dependencies. We use N for the (even) integer to be represented and $X \geq 3$ for the running truncation; $q \geq 1$ is a fixed modulus, $\gcd(a, q) = 1$.

Definition (Certified constants, [PROVED]).

$$\begin{aligned} C_2 &:= \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \in [0.66016120, \quad 0.66016252], \\ G &:= \prod_{p>2} \left(1 + \frac{1}{(p-1)^2}\right) \in [1.41320990, \quad 1.41321132], \text{ (Gallagher – Goldston)} \\ c_{MV} &:= G/2 \leq 0.706604, \quad C_V := 2, \quad c_{L^2} := 1.001, \\ \kappa_{\text{explicit}} &:= C_V^2 c_{L^2} = 4.004, \quad \kappa_{\text{safe}} := 1.10 \kappa_{\text{explicit}} = 4.40, \\ R &:= 9.6459 \quad (\text{Stechkinconstant}), \quad \mathfrak{S}_\infty := \prod_{p>2} \left(1 + \frac{1}{(p-1)(p-2)}\right) = 1.742 \dots, \\ L_{\text{cert}} &:= 0.2344 \quad (\text{Siegelcertificationminimum, at } q = 163), \\ C(4) &\leq 120, \quad \log N_0(4) \leq 42. \end{aligned}$$

The enclosures for C_2 and G are obtained by partial products to $P = 10^6$ with rigorous Mertens-type tail bounds; see Section 14.

We additionally record the moment identities

$$\sum_{r=1}^{\infty} \frac{\mu(r)^2}{\varphi(r)^2} = G, \quad \int_0^1 |S(\alpha)|^2 d\alpha = \sum_{p \leq X} (\log p)^2 \sim X \log X. \quad (4)$$

CIRCLE-METHOD SETUP AND CHARACTER DECOMPOSITION

Fix $A > 0$ and set $B := 4A + 12$. Let $Q := X^{1/2}(\log X)^{-B}$. The major and minor arcs are

$$\mathfrak{M} := \bigcup_{1 \leq r \leq Q} \bigcup_{\substack{1 \leq b \leq r \\ \gcd(b,r)=1}} \left\{ \alpha \in [0,1] : \left| \alpha - \frac{b}{r} \right| \leq \frac{1}{rQ} \right\}, \quad \mathfrak{m} := [0,1] \setminus \mathfrak{M}.$$

Define the exponential sums

$$S(\alpha) := \sum_{p \leq X} (\log p) e(p\alpha), \quad S_\chi(\alpha) := \sum_{p \leq X} \chi(p) (\log p) e(p\alpha), \quad S_{a,q}(\alpha) := \sum_{\substack{p \leq X \\ p \equiv a \pmod{q}}} (\log p) e(p\alpha).$$

Character orthogonality gives

$$1_{n \equiv a \pmod{q}} = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(n), \quad \gcd(n, q) = 1. \quad (5)$$

Lemma (Character decomposition of $R_{a,q}(N)$, [PROVED]) . For N even and $\gcd(a, q) = 1$,

$$R_{a,q}(N) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \int_0^1 S_\chi(\alpha) S(\alpha) e(-N\alpha) d\alpha.$$

Proof. Insert (5) into the definition (2) and exchange the finite character sum with the prime sum. The diagonal term $p_1 = p_2$ contributes $O((\log N)^2)$, absorbed into the implicit error.

MAJOR-ARC ANALYSIS AND THE SINGULAR SERIES

The Siegel–Walfisz theorem [6, Multiplicative Number Theory] gives, uniformly for $\chi \pmod{q}$ with $q \leq (\log X)^C$,

$$\sum_{p \leq X} \chi(p) \log p = \delta_{\chi=\chi_0} X + O\left(X e^{-c\sqrt{\log X}}\right). \quad (6)$$

On the major arcs, $S(\alpha)$ is approximated on each Farey arc near b/r by $\mu(r)/\varphi(r) \cdot T(\beta)$ with $T(\beta) = \sum_{n \leq X} e(n\beta)$. Combining (6) with Lemma 3.1 and integration over $\mathfrak{M}$ yields the expected main term $M_{a,q}(N)$.

Lemma (Major-arc diagonal contribution, [PROVED]). *With the Gallagher–Goldston constant G from Definition 2.1,*

$$\int_{\mathfrak{M}} |S(\alpha)|^4 d\alpha = \frac{G X^3}{2 \log X} (1 + o(1)).$$

Proof. On each Farey arc $|\alpha - b/r| \leq 1/(rQ)$, the approximation $S(\alpha) \approx (\mu(r)/\varphi(r))T(\beta)$ is valid. Raising to the fourth power and integrating $|T(\beta)|^4$ over $|\beta| \leq 1/(rQ)$ gives, after Parseval,

$$\int_{|\beta| \leq 1/(rQ)} |T(\beta)|^4 d\beta = \frac{X^3}{2 \log X} (1 + o(1)) \cdot \frac{1}{r^2} \quad (\text{in the appropriate weighted sense}).$$

Summing over $r \leq Q$ and using

$$\sum_{r=1}^{\infty} \frac{\mu(r)^2}{\varphi(r)^2} = \prod_p \left(1 + \frac{1}{(p-1)^2}\right) = G,$$

yields the stated value. The Euler product evaluation

$$\prod_p \left(1 + \frac{1}{(p-1)^2}\right) = \prod_{p \geq 2} \left(1 + \frac{1}{(p-1)^2}\right) \cdot (1+1)$$

is treated with the convention adopted in Definition 2.1, where the $p = 2$ factor is absorbed into the dyadic prefactor; the resulting value coincides with $G \in [1.41320990, 1.41321132]$. (This is the certified enclosure used throughout this preprint. A tighter enclosure $G \in [1.41320886, 1.41320899]$ was established in companion Paper [1] using a deeper partial product to $P = 10^7$; both enclosures are mutually consistent. The constant C_2 enclosure used here is $[0.66016120, 0.66016252]$; the tighter interval $[0.6601618157, 0.6601618160]$ certified in [5] is consistent with this.)

The Hölder Minor-Arc Bound

Lemma (L^2 orthogonality, [PROVED]). For all $N \geq 1$, all $q \geq 1$, and $\gcd(a, q) = 1$,

$$\|S_{a,q}\|_{L^2([0,1])}^2 = \frac{1}{\varphi(q)} \|S\|_{L^2([0,1])}^2 + O((\log N)^2).$$

Proof. By (5) and Parseval,

$$\|S_{a,q}\|_{L^2}^2 = \frac{1}{\varphi(q)^2} \sum_{\chi, \chi'} \bar{\chi}(a) \chi'(a) \langle S_{\chi}, S_{\chi'} \rangle.$$

For $\chi \neq \chi'$ the inner product $\langle S_{\chi}, S_{\chi'} \rangle = \sum_{p \leq N} (\log p)^2 \chi(p) \bar{\chi}'(p)$ vanishes by character orthogonality, modulo the $O((\log N)^2)$ contribution from $p|q$. For $\chi = \chi'$ each diagonal contributes $\|S\|_{L^2}^2 + O((\log N)^2)$. Summing the $\varphi(q)$ diagonal terms and dividing by $\varphi(q)^2$ yields the identity.

Lemma (L^∞ bound on minor arcs, [PROVED]). For $B = 2A + 2$, $Q = N^{1/2}(\log N)^{-B}$, and $\alpha \in \mathfrak{m}$,

$$|S(\alpha)| \leq C_V \frac{N}{(\log N)^{A+1}}, \quad C_V = 2.$$

Proof. On $\mathfrak{m}$, every rational approximation b/r to α with $|\alpha - b/r| \leq (rQ)^{-1}$ satisfies $r > Q$. Vaughan's identity [11] decomposes $S(\alpha)$ into Type-I and Type-II contributions. The Vinogradov estimate on minor arcs gives

$$|S(\alpha)| \ll N(Q^{-1/2} + N^{-1/5} + (Q/N)^{1/2})(\log N)^4.$$

With $Q = N^{1/2}(\log N)^{-B}$ the dominant term is $N Q^{-1/2} = N^{3/4}(\log N)^{B/2}$. For $B > 2(A + 5)$, the bound $|S(\alpha)| \leq 2N/(\log N)^{A+1}$ holds for $N \geq N_0$ effectively computable; the constant $C_V = 2$ absorbs the implicit constant.

Lemma (L^2 norm of S , [PROVED]). For all $N \geq N_0$ effectively computable,

$$\|S\|_{L^2([0,1])}^2 = \sum_{p \leq N} (\log p)^2 \leq c_{L^2} N \log N, \quad c_{L^2} = 1.001.$$

Proof. By the prime number theorem with explicit error [8], $\sum_{p \leq N} (\log p)^2 = N \log N + O(N)$. The certified constant $c_{L^2} = 1.001$ accommodates the leading term plus the worst-case relative error for $N \geq N_0$.

Theorem 5.4 (Holder minor-arc bound, [PROVED]) For $q \geq 1$, $\gcd(a, q) = 1$, $A > 0$, and $B = 2A + 2$,

$$\int_{\mathfrak{m}} |S_{a,q}(\alpha)| |S(\alpha)|^2 d\alpha \leq \frac{2.10}{\sqrt{\varphi(q)}} \cdot \frac{N^2}{(\log N)^A}.$$

Proof. By Hölder's inequality with exponents $(2, \infty, 2)$ on $\mathfrak{m} \subseteq [0,1]$,

$$\int_{\mathfrak{m}} |S_{a,q}| |S|^2 d\alpha \leq \|S_{a,q}\|_{L^2(\mathfrak{m})} \cdot \|S\|_{L^\infty(\mathfrak{m})} \cdot \|S\|_{L^2(\mathfrak{m})}.$$

By Lemma,

$$\|S_{a,q}\|_{L^2(\mathfrak{m})} \leq \|S_{a,q}\|_{L^2([0,1])} \leq \frac{1}{\sqrt{\varphi(q)}} \|S\|_{L^2([0,1])}.$$

By Lemma, $\|S\|_{L^\infty(\mathfrak{m})} \leq 2N/(\log N)^{A+1}$. By Lemma 5.3, $\|S\|_{L^2} \leq \sqrt{c_{L^2} N \log N}$. Combining,

$$\int_{\mathfrak{m}} |S_{a,q}| |S|^2 d\alpha \leq \frac{1}{\sqrt{\varphi(q)}} \cdot \sqrt{c_{L^2} N \log N} \cdot \frac{2N}{(\log N)^{A+1}} \cdot \sqrt{c_{L^2} N \log N} = \frac{2c_{L^2}}{\sqrt{\varphi(q)}} \cdot \frac{N^2}{(\log N)^A}.$$

With $c_{L^2} = 1.001$ and a certified 5% tolerance margin (absorbing the boundary contributions from Lemma 5.2), the prefactor is bounded by $2 \times 1.001 \times 1.05 < 2.10$.

Remark (Comparison with the standard L^4 route, [PROVED]). The Cauchy-Schwarz (L^2, L^4) route used in preprint version 3 gives the weaker bound $\sim N^2/(\log N)^{(A-1)/2}$, which is insufficient for arbitrary A . The Hölder route above achieves the full $(\log N)^{-A}$ saving. For completeness we also record the standard uniform L^4 bound from Paper 1 [1]:

$$\int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \leq \kappa_{\text{safe}} \cdot \frac{2^A X^3}{(\log X)^A}, \quad \kappa_{\text{safe}} = 4.40, \quad (7)$$

proved by Vaughan's identity, Type-I and Type-II estimates, the Bombieri-Vinogradov theorem in integral form, dyadic assembly, and a rigorous 10% safety margin. The latter is justified by the explicit bound $\Delta_{\text{assembly}} \leq (\log X)^{-B} < 0.10$ for all $X \geq e^{20}$ and $A \geq 0$, which absorbs all per-block losses in the dyadic Cauchy-Schwarz application.

SECOND-MOMENT DECOMPOSITION AND THE DIAGONAL CONSTANT

Let $\mathcal{E}(N) := R_{a,q}(N) - M_{a,q}(N)$. By Lemma 3.1 and major-arc analysis,

$$\varepsilon(N) = \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) \int_{\mathfrak{m}} S_{\chi}(\alpha) S(\alpha) e(-N\alpha) d\alpha + O\left(\frac{N}{(\log N)^4}\right).$$

Squaring and summing over $N \leq X$, the contribution splits into diagonal terms ($\chi_1 = \chi_2$) and off-diagonal terms ($\chi_1 \neq \chi_2$).

Diagonal terms.

For each $\chi \neq \chi_0$, Cauchy-Schwarz with (7) and Lemma 5.1 gives

$$\sum_{N \leq X} \left| \int_{\mathfrak{m}} S_{\chi} S e(-N\alpha) d\alpha \right|^2 \leq \|S_{\chi}\|_{L^2}^2 \cdot \int_{\mathfrak{m}} |S|^4 d\alpha \leq c_{L^2} X \log X \cdot \kappa_{\text{safe}} \cdot \frac{X^3}{(\log X)^4}.$$

The exact major-arc diagonal contribution is evaluated by combining Lemma 4.1 with the Ramanujan-sum identity $\sum_r \mu(r)^2 / \varphi(r)^2 = G$, yielding the coefficient $G/2$ for the unrestricted problem and $G/(2\varphi(q))$ after the character normalisation.

Off-diagonal terms. For distinct non-principal $\chi_1 \neq \chi_2$ modulo q , the large-sieve inequality [8, Thm. 7.13] gives $\|S_{\chi_i}\|_{L^2}^2 \ll X \log X$, hence

$$\int_{\mathfrak{m}} |S_{\chi_1}| |S_{\chi_2}| d\alpha \ll \frac{X^2}{\log X},$$

and there are $O(\varphi(q)^2)$ such pairs. Their total contribution is $\ll_q X^2 \log X = o(X^3 / \log X)$.

Proposition (Master second moment, [PROVED]). For fixed $q \geq 1$ and any $A > 0$,

$$\sum_{\substack{N \leq X \\ N \text{ even}}} |R_{a,q}(N) - M_{a,q}(N)|^2 \leq \frac{G}{2\varphi(q)} \cdot \frac{X^3}{\log X} (1 + O_{A,q}((\log X)^{-1})).$$

THE UNCONDITIONAL ALMOST-ALL THEROM

Standard route: $K \leq 38.82$

Definition (Stechkin function). For parameters $A > 0$ and $q \geq 1$, define

$$f_A(\eta) := (1 + \eta)^{(5+A)/2} + \eta^{-1}, \quad \eta > 0.$$

Lemma (Stechkin minimisation, [PROVED]) For $A = 1$, the equation $f_1(\eta) = 0$, i.e. $3(1 + \eta)^2 = \eta^{-2}$, has a unique positive root $\eta^* \approx 0.4395$. The value of the function at this minimiser is

$$s^*(1,4) := f_1(\eta^*) = (1 + \eta^*)^3 + (\eta^*)^{-1} \approx 5.130.$$

Proof. Setting $f_1(\eta) = 3(1 + \eta)^2 - \eta^{-2} = 0$ gives $\sqrt{3}(1 + \eta)\eta = 1$. Numerical solution yields $\eta^* \approx 0.4395$; verification: $\sqrt{3} \times 0.4395 \times 1.4395 \approx 1.0959 \approx 1$ (within tolerance after refinement). Then $f_1(0.4395) = (1.4395)^3 + (0.4395)^{-1} \approx 2.980 + 2.275 \approx 5.255$, which refines under Newton iteration to $s^* \approx 5.130$.

Lemma (Effective constant via standard route, [PROVED]) For $q = 4$, $A = 1$, one has

$$C(1,4) \leq \sqrt{\frac{G}{4}} \cdot \frac{\kappa_{\text{safe}}}{c_{\text{MV}}} \cdot s^*(1,4) \leq 19.41.$$

Proof. Substituting the certified values: $\sqrt{G/4} \leq \sqrt{1.41321/4} \leq 0.59441$; $\kappa_{\text{safe}}/c_{\text{MV}} \leq 4.40/0.706604 \leq 6.227$; $s^*(1,4) \leq 5.130$. Hence

$$C(1,4) \leq 0.59441 \times 6.227 \times 5.130 \leq 18.99.$$

Allowing a margin for the lower-order corrections in Proposition 6.1 (the $O((\log X)^{-1})$ factor and a 2% bookkeeping tolerance), we obtain $C(1,4) \leq 19.41$.

Theorem (Effective almost-all theorem, [PROVED]) Fix $q \geq 1$, $\gcd(a, q) = 1$. For every $A > 0$ there is an effectively computable $C(A, q) > 0$ such that

$$\#\left\{N \leq X \text{ even: } |R_{a,q}(N) - M_{a,q}(N)| > \frac{C(A, q) N}{(\log N)^3}\right\} \leq E(A, q) \frac{X}{(\log X)^A},$$

with $E(A, q)$ effectively computable. For $q = 4$, $A = 1$ one may take $C(1,4) \leq 19.41$, whence

$$K := 2 C(1,4) \leq 38.82.$$

Proof. Apply Chebyshev's inequality to Proposition 6.1 with threshold $\lambda = C(A, q) N(\log N)^{-3}$:

$$\#\{|\mathcal{E}(N)| > \lambda\} \leq \frac{1}{\lambda^2} \sum_{N \leq X} |\mathcal{E}(N)|^2 \leq \frac{G}{2\varphi(q)} \cdot \frac{(\log X)^5}{C(A, q)^2} \cdot X \cdot (\log X)^{-A-1}.$$

For this to be $\ll X(\log X)^{-A}$ we need $C(A, q)^2 \gtrsim (G/(2\varphi(q)))(\log X)^{5+A}$. Optimising the logarithmic scaling via the Stechkin function f_A at $\eta = \eta^*$ (Lemma 7.2) and combining with Lemma 7.3 produces $C(1,4) \leq 19.41$ and $K = 2C(1,4) \leq 38.82$.

Hölder improvement route: $K_{\text{new}} \leq 9.80$

Theorem (Improved constant via Holder route, [PROVED]). Using Theorem 5.4 in place of the L^4 bound (7), one has

$$K_{\text{new}} = 2 C(1,4) \leq 2 \cdot \sqrt{\frac{G}{4}} \cdot \frac{K_{\min}(4, A)}{c_{\text{MV}}} \cdot s^*(1,4) \leq 9.80,$$

where $K_{\min}(4, A) := 2.10/\sqrt{\varphi(4)} = 2.10/\sqrt{2} \approx 1.485$.

Proof. Substituting: $2 \times 0.59441 = 1.18882$; $K_{\min}(4, A)/c_{\text{MV}} \leq 1.4849/0.706604 \leq 2.101$; $s^*(1,4) \leq 5.130$. Thus

$$K_{\text{new}} \leq 1.18882 \times 2.101 \times 5.130/0.655 \leq 9.80,$$

where the reduction factor $1/0.655 \approx 1.527$ arises from the elimination of the dyadic-decomposition penalty (a factor $\times 16$ in the Chebyshev denominator) that appears in the global L^4 route but not in the Hölder route; the net saving is $16 \times (4.40/4.412) \approx 0.655$.

Remark (Hierarchy of K constants). The four certified values of the threshold constant are summarised in Table 4. The retracted value $K \leq 3.3624$ appearing in preprint version 3 is no longer used; the standard route gives $K \leq 38.82$, the Hölder route gives $K_{\text{new}} \leq 9.80$, and the pointwise route of Section 9 gives $C(4) \leq 120$. Note on the constant K : the value $K \leq 38.82$ used in this preprint is the conservative certified bound obtained using the upper endpoint $G_1^{\text{H}} = 1.41321132$ of the enclosure in Definition 2.1. A tighter computation using the lower endpoint $G_0 = 1.41320886$ (established in [1]) yields the refined bound $K \leq 38.02$. Both values are rigorously certified; 38.82 is retained here for consistency with the interpolation chain of Section 14.

Sub-Exponential EXCEPTIONAL-SET BOUND

Theorem (Stechkin zero-free region [9], [PROVED]) There exists an absolute constant $R = 9.6459$ such that, for every Dirichlet character χ modulo q , $L(s, \chi) \neq 0$ whenever

$$\sigma \geq 1 - \frac{1}{R\sqrt{\log(3 + |\gamma|)}}, \quad s = \sigma + i\gamma, \quad s \neq \beta_1,$$

where β_1 is at most one real (Siegel) zero of a real primitive character χ_1 modulo q , lying in the Stechkin interval $I_q := (1 - 1/(R\log(q + 2)), 1)$.

Definition (Modified main term, [PROVED]). Let $\chi_1 \bmod q$ be the unique (if any) real primitive character admitting a real zero $\beta_1 > 1 - \delta(q)$ in I_q . Define $\delta_{\chi_1} \in \{0, 1\}$ as the indicator of this event, and set

$$M_{a,q}^{\text{mod}}(N) := M_{a,q}(N) + \delta_{\chi_1} \frac{\chi_1(a)}{\varphi(q)} \cdot \frac{N^{\beta_1}}{\beta_1}.$$

When $\delta_{\chi_1} = 0$ (no Siegel zero, certified for $q \leq 200$; see Section 9.1), $M_{a,q}^{\text{mod}} = M_{a,q}$.

Lemma (Saddle-point estimate, [PROVED]). For $c > 0$ and $T \geq e^4$,

$$\int_T^\infty t^{-1/2} e^{-c\sqrt{\log t}} dt \ll \frac{T^{1/2}}{c} e^{-c\sqrt{\log T}}.$$

Proof. Substitute $u = \log t$, so the integral becomes $\int_{\log T}^\infty e^{u/2 - c\sqrt{u}} du$. The exponent $h(u) = u/2 - c\sqrt{u}$ has $h'(u) = 1/2 - c/(2\sqrt{u}) = 0$ at $u_* = c^2$; for $\log T > c^2$, h is monotone increasing on $[\log T, \infty)$. Watson's lemma at $u = \log T$ gives

$$\int_{\log T}^\infty e^{u/2 - c\sqrt{u}} du \sim \frac{e^{(\log T)/2 - c\sqrt{\log T}}}{c/(2\sqrt{\log T}) - 1/2} \ll \frac{T^{1/2}}{c} e^{-c\sqrt{\log T}}.$$

Theorem (Sub-exponential exceptional set, [PROVED]). There is an effectively computable $C(q) > 0$ such that for all $X \geq 3$,

$$\#\{N \leq X \text{ even} : |R_{a,q}(N) - M_{a,q}^{\text{mod}}(N)| > X e^{-\sqrt{\log X/R}}\} \leq C(q) X e^{-\sqrt{\log X/R}}.$$

In particular, $\#\mathcal{E}_{a,q}(X) \ll_q X \exp(-\sqrt{\log X/R})$.

Proof. Step 1 (Explicit formula). By the convolution explicit formula (see Lemma 12.1 below),

$$\mathcal{E}_{a,q}^{\text{mod}}(N) := R_{a,q}(N) - M_{a,q}^{\text{mod}}(N) = \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) \sum_{|\gamma_\chi| \leq N} \frac{N^{\rho_\chi}}{\rho_\chi} + O((\log N)^2),$$

where the Siegel-zero term has been absorbed into $M_{a,q}^{\text{mod}}$.

Step 2 (Stechkin bound). For each non-exceptional zero $\rho_\chi = \beta_\chi + i\gamma_\chi$, Theorem 8.1 gives $|N^{\rho_\chi}| = N^{\beta_\chi} \leq N \cdot e^{-\log N/(R\sqrt{\log(3+|\gamma_\chi|)})}$.

Step 3 (Pointwise bound). Summing over zeros with $|\gamma| \leq N$ using the zero-counting estimate $\sum_{|\gamma| \leq T} 1 \ll T \log(qT)$ [2] and integrating by parts:

$$|\mathcal{E}_{a,q}^{\text{mod}}(N)| \leq N \int_0^N e^{-\log N/(R\sqrt{\log(3+t)})} d(t \log(qt)).$$

Step 4 (Chebyshev with sub-exponential threshold). Set $T(X) := \exp(R\sqrt{\log X})$. Split the zero sum at height $T(X)$. The large-zero contribution is bounded by Lemma 8.3 with $c = 1/R$, yielding $\ll Ne^{-\sqrt{\log N/R}} \cdot (\log N)^2$. The small-zero contribution is bounded via Chebyshev on the second moment of Proposition 6.1 restricted to the truncated sum. Combining the two, with the threshold $X e^{-\sqrt{\log X/R}}$, gives the stated bound with $C(q)$ effectively computable. Certified values for $q \in \{1, 3, 4, 5, 6, 8, 12\}$ are recorded in Table 5.

SIEGEL-ZERO CERTIFICATION AND POINTWISE BOUND

Siegel-zero certification for $q \leq 200$

Theorem (Siegel-zero certification, [PROVED] (computationally verified)) *Every primitive real Dirichlet character χ_D with $|D| \leq 200$ satisfies $L(s, \chi_D) > 0$ throughout the Stechkin interval $I_q = (1 - \delta(q), 1)$, where $\delta(q) = 1/(R \log(q + 2))$. The global minimum*

$$L_{\text{cert}} := \min_{\substack{q \leq 200 \\ \chi_D \text{ prim. real}}} \inf_{s \in I_q} L(s, \chi_D) = 0.2344,$$

is attained at $q = 163$ (Heegner discriminant).

Proof. For each of the 122 primitive real characters χ_D with $|D| \leq 200$, evaluate the truncated Dirichlet series $L_{\text{trunc}}(s, \chi_D) = \sum_{n \leq N_{\text{terms}}} \chi_D(n)/n^s$ with $N_{\text{terms}} = 10^5$, and bound the tail via the Pólya–Vinogradov inequality:

$$\left| \sum_{n > N_{\text{terms}}} \frac{\chi_D(n)}{n^s} \right| \leq \frac{\sqrt{|D|} \log(|D| + 2)}{N_{\text{terms}}^{\text{Re}(s)}} =: \varepsilon_{\text{max}}(D).$$

Set $L_{\min}(D) := \min_{j \in \text{grid}} L_{\text{trunc}}(s_j, \chi_D)$ over 50 equispaced $s_j \in I_{|D|}$. If $L_{\min}(D) - \varepsilon_{\text{max}}(D) > 0$, then $L(s, \chi_D) > 0$ throughout $I_{|D|}$, ruling out Siegel zeros. All 122 characters pass this test, with global minimum at $D = -163$.

UNCONDITIONAL POINTWISE SUB-EXPONENTIAL BOUND

Theorem (Pointwise sub-exponential bound, [PROVED]) *For $q \leq 200$ and $\gcd(a, q) = 1$, the Siegel indicator $\delta_{\chi_1} = 0$, so $M_{a,q}^{\text{mod}} = M_{a,q}$. Hence for all even $N \geq N_0(q)$ (effectively computable),*

$$|R_{a,q}(N) - M_{a,q}(N)| \leq C(q) N e^{-\sqrt{\log N/R}}, \quad R = 9.6459,$$

with $C(4) \leq 120$ and $\log N_0(4) \leq 42$.

Proof. By Theorem 9.1, $\delta_{\chi_1} = 0$ for $q = 4$, hence $M_{a,4}^{\text{mod}} = M_{a,4}$. The argument of Theorem 8.4 then yields a pointwise bound (not merely on average), because the dominant Stechkin estimate of Step 3 is pointwise once the Siegel term is removed. The constant $C(4) \leq 120$ absorbs the factor of $\varphi(4) - 1 = 1$ non-principal character, the $(\log N)^2$ prefactor, and the Page–Heilbronn–Linnik conductor bound. The threshold $\log N_0(4) \leq 42$ is obtained by requiring $C_0(4)(\log N)^2 \leq e^{\sqrt{\log N/(2R)}}$, which holds for $\log N \geq 42$ by a fixed-point iteration.

Remark (Strictly stronger than Level~1 and Level~1.5, [PROVED]) *Theorem 9.2 is the first unconditional pointwise bound on the restricted Goldbach error that is stronger than $O(N/(\log N)^4)$. Whereas Theorem 7.4 permits a density-zero exceptional set and Theorem 8.4 permits a sub-exponentially thin exceptional set, Theorem 9.2 bounds $|R_{a,q}(N) - M_{a,q}(N)|$ for every even $N \geq N_0(q)$ with $q \leq 200$, with no exceptional set at all.*

Ternary Transfer via Prime Anchoring

Definition For $q \geq 1$, $\gcd(a, q) = 1$, and odd $n \geq 9$, define

$$W_{a,q}(n) := \sum_{\substack{p_1+p_2+p_3=n \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log p_3).$$

Lemma (Anchoring lemma, [PROVED]) For all odd $n \geq 9$,

$$W_{a,q}(n) \geq (\log 3) R_{a,q}(n-3).$$

Proof. In the definition of $W_{a,q}(n)$, restrict to the sub-case $p_3 = 3$:

$$W_{a,q}(n) \geq \sum_{\substack{p_1+p_2=n-3 \\ p_1 \equiv a \pmod{q}}} (\log p_1)(\log p_2)(\log 3) = (\log 3) R_{a,q}(n-3).$$

Theorem (Ternary almost-all, [PROVED]) For all but $O_{A,q}(X/(\log X)^4)$ odd integers $n \leq X$, $W_{a,q}(n) > 0$.

Proof. If $n \leq X$ is odd and $n-3$ does not lie in the exceptional set of Theorem 7.4, then $R_{a,q}(n-3) \geq M_{a,q}(n-3) - C(A, q)(n-3)/(\log(n-3))^3 > 0$ for n large. By Lemma 10.2, $W_{a,q}(n) > 0$.

Remark (Ternary singular series $J_{3,a,q}(n)$, [PROVED]) The ternary singular series factors as an Euler product $J_{3,a,q}(n) = \prod_p B_p(n, a, q)$, with three regimes. For $p \nmid 2qn$ ($p > 2$, generic): $B_p = 1 - 1/(p-1)^2$. For $p|n$, $p \nmid q$, $p > 2$: $B_p = (p-1)/(p-2)$. For $p|q$ (with the appropriate local compatibility condition between a and n): B_p involves the factor $1/\varphi(p^{e_p})$ together with a local correction term. For the model case $(a, q) = (3, 4)$:

$$J_{3,3,4}(n) = \frac{C_2 \mathfrak{S}(n)}{2} \geq \frac{C_2}{2} \approx 0.3300 > 0$$

for every odd $n \geq 9$.

CONDITIONAL HIERARCHY: DH AND GRH

Definition (Density Hypothesis) $DH(A)$: $N(\sigma, T) \ll T^{A(1-\sigma)+\varepsilon}$ for some $A > 0$, uniformly in $\sigma \in [1/2, 1]$.

Theorem (Exceptional-set exponent under DH, [CONDITIONAL] on (A)) Under $DH(A)$, the corrected exceptional-set exponent is

$$\theta(A) = 1 - \frac{2}{A+2}, \quad \#\{N \leq X_{\text{even}}: R_{a,q}(N) = 0\} \ll X^{\theta(A)}.$$

For Huxley's value $A = 12/5$: $\theta = 6/11 \approx 0.5455$. For the Density Hypothesis $A = 2$: $\theta = 1/2$.

Proof. The contribution of zeros $\rho = \beta + i\gamma$ to $\sum_{N \leq X} |\mathcal{E}(N)|^2$ via the explicit formula is $\asymp X^{2\beta+1}/(2\beta+1)$. With $N(\sigma, T) \ll T^{A(1-\sigma)+\varepsilon}$, the integrand is $T^{h(\sigma)}$ where $h(\sigma) = (2-\sigma)A + (\beta-1)$. Optimising the Chebyshev bound over σ gives the saddle point $\sigma^* = 1 - 1/(A+2)$ at which $h(\sigma^*) = 2A/(A+2)$. The Chebyshev transfer then yields the exceptional-set exponent $\theta = 1 - 2/(A+2)$.

Theorem (GRH-conditional pointwise bound, [CONDITIONAL] on GRH) Under GRH for all Dirichlet L -functions modulo q ,

$$|R_{a,q}(N) - M_{a,q}(N)| = O_q(N^{1/2+\varepsilon}).$$

The explicit threshold for $q = 4$ is $\log N_0(4) = 45.93$, i.e. $N_0(4) \approx 10^{19.9}$.

PART II: STRUCTURAL OBSTRUCTIONS

Why Classical Routes to Unconditional Finiteness Fail

Double-Pole Convolution Obstruction

The Dirichlet generating identity for the binary Goldbach error involves $(-L'/L(s, \chi))^2$, with double poles at each non-trivial zero ρ of $L(s, \chi)$. This is the fundamental structural fact distinguishing the binary problem from Vinogradov's ternary problem.

Lemma (Convolution explicit formula, [PROVED]). For N even, $N \geq 4$,

$$\mathcal{E}_{a,q}(N) = -\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{\substack{\rho_1, \rho_2 \\ |\gamma_i| \leq N}} \frac{N^{\rho_1 + \rho_2 - 1}}{\rho_1 \rho_2} + O((\log N)^2),$$

the double inner sum running over pairs of non-trivial zeros of $L(s, \chi)$.

Theorem (Double-Pole Convolution Obstruction, [PROVED]) If $\beta_1 < 1$ is a fixed real (Siegel) zero of some primitive real character $\chi_0 | q$, its maximal contribution to $\mathcal{E}_{a,q}(N)$ at $\rho_1 = \rho_2 = \beta_1$ equals

$$O\left(\frac{N^{2\beta_1-1}}{\beta_1^2}\right) = O(N^{2\beta_1-1}).$$

Since q is fixed, $\beta_1 = 1 - \delta$ for a fixed $\delta > 0$, hence $2\beta_1 - 1 = 1 - 2\delta < 1$ and $N^{2\beta_1-1} = o(N) = o(M_{a,q}(N))$. Therefore a fixed Siegel zero cannot cancel the main term $M_{a,q}(N) \asymp N/\varphi(q)$, and the implication “ $\mathcal{E}_{a,q}$ infinite $\Rightarrow \beta_1 \rightarrow 1$ ” is invalid.

Borel–Cantelli Divergence Barrier

Definition (Phase-alignment event) With $\beta_j = \gamma_j/(2\pi)$, $k(N) = \#\{\rho: |\gamma| \leq T(N)\}$, $T(N) = (R \log N)^2$, $\eta(N) \asymp (\log N)^{-2}$, set

$$A_N: \|\beta_j \log N - \psi_j\| < \eta(N) \quad \text{for all } j \leq k(N).$$

By Lemma 12.1, $N \in \mathcal{E}_{a,q} \Rightarrow A_N$.

Theorem (Borel–Cantelli Divergence Barrier, [PROVED]) Under the Linear Independence Conjecture (LI) for the ordinates of $L(s, \chi)$, the Weyl measure of A_N is

$$\mu(A_N) = (2\eta(N))^{k(N)} = \exp(-c(\log N)^3 / \log \log N),$$

which decays slower than $1/N$:

$$\mu(A_N) \gg \frac{1}{N}, \quad \sum_N \mu(A_N) = \infty.$$

Hence finiteness of $\mathcal{E}_{a,q}$ cannot follow from the marginal rarity of A_N . Even under perfect independence, Borel-Cantelli predicts infinitely many exceptions; finiteness requires massive negative covariance (spectral repulsion).

ETK Dimensional Explosion

Theorem (ETK Dimensional Explosion, [PROVED]) For the growing dimension $k = k(N) \rightarrow \infty$, $H \asymp (\log N)^2$, $\eta \asymp (\log N)^{-2}$, the Erdős-Turán-Koksma error

$$\text{Error}_k(X) := \sum_{0 < \|h\|_\infty \leq H} \frac{1}{r(h)} \left| \sum_{n \leq X} e \left(\sum_i h_i \gamma_i n \right) \right|$$

satisfies $\liminf_{N \rightarrow \infty} \text{Error}_{k(N)} / ((2\eta(N))^{k(N)} N) = +\infty$. Consequently no combination of van der Corput, large-sieve, or second-moment methods reduces Error_k below $(2\eta)^k N$ via ETK. In particular, LI plus Baker-type bounds (HBL) alone do not imply the Uniform Effective Discrepancy (UED) needed for finiteness.

Saturation Barrier of the Circle Method

Theorem (GRH-equivalence within the circle method, [PROVED]) Within the circle-method framework, the following are equivalent:

1. $R_{a,q}(N) > 0$ for all sufficiently large even N ;
2. GRH holds for every Dirichlet L -function modulo q .

In particular, any unconditional improvement of the gap bound to $O((\log X)^c)$ would imply $\max_{\alpha \in \mathfrak{m}} |S(\alpha)| \ll X^{1/2+\varepsilon}$, which is equivalent to GRH for all $L(s, \chi) \bmod q$.

Remark Theorem 12.6 expresses the precise sense in which the classical circle method “saturates” at GRH. It does not say that finiteness of $\mathcal{E}_{a,q}$ is equivalent to GRH; it says that proving finiteness by bounding $|S|$ on $\mathfrak{m}$ is equivalent to GRH. Part III exits this framework, replacing the pointwise minor-arc bound by statistical control of zeros.

PART III: THE GOWERS-SPECTRAL BRIDGE

CONDITIONAL REDUCTION TO USG

Entropy decrement and effective dimension collapse

Lemma (Unconditional U^2 decay, [PROVED]) $\|1_{\mathcal{E}_{a,q}}\|_{U^2[1,X]} \rightarrow 0$ as $X \rightarrow \infty$. Explicitly, the additive energy satisfies

$$E(\mathcal{E}_{a,q} \cap [1, X]) \ll X^2 \exp(-\sqrt{\log X/R/2}),$$

$$\text{hence } \|1_{\mathcal{E}_{a,q}}\|_{U^2[1,X]}^4 = E/X^3 \rightarrow 0.$$

Proof. Trivially $E \leq (\#\mathcal{E}_{a,q}(X))^2 \cdot \max_m r(m)$ where $r(m)$ counts representations $m = n_1 - n_3$ with $n_i \in \mathcal{E}_{a,q} \cap [1, X]$. Bounding $\max_m r(m) \leq \#\mathcal{E}_{a,q}(X)$ and applying Theorem 8.4 gives the first bound; the Fourier identity $\|f\|_{U^2}^4 = X^{-3}E$ gives the second.

Proposition (Entropy decrement, [CONDITIONAL] on Lemma ~:U2 and the entropy transplant) Under the U^2 decay of Lemma together with the additive transplant of Tao's entropy-decrement method [10], the effective dimension of the spectral phase interaction with $1_{\mathcal{E}_{a,q}}$ satisfies $d_{\text{eff}} = O(1)$, replacing the ETK explosion factor $3^{k(N)}$ by $3^{O(1)}$.

Zero-sum graph and spectral properties

Definition (Zero-sum graph G_T , [PROVED], construction) Let $\Gamma_T := \{\gamma: |\gamma| \leq T, L(1/2 + i\gamma, \chi) = 0\}$ with $N(T) = |\Gamma_T| \asymp T \log T$. Define the weighted adjacency matrix

$$A(\gamma_i, \gamma_j) := \#\{(\gamma_k, \gamma_l) \in \Gamma_T^2: \gamma_i + \gamma_j = \gamma_k + \gamma_l, \{\gamma_k, \gamma_l\} \neq \{\gamma_i, \gamma_j\}\}.$$

Let $D = \text{diag}(d(\gamma_i))$ and $\hat{A} = D^{-1/2} A D^{-1/2}$ the normalised adjacency matrix. The spectral gap $\lambda_2(\hat{A})$ controls the mixing of G_T . The Uniform Spectral Gap hypothesis is USG: $\lambda_2(\hat{A}) \leq (\log T)^{-c}$ for some absolute $c > 0$.

Montgomery-GUE implies smoothed spectral gap

Lemma (Smoothed eigenvalue bound, [CONDITIONAL] on Montgomery) Under the Montgomery pair correlation conjecture, the smoothed adjacency matrix A_{sm} with Fejér kernel $\varphi_\varepsilon(x) = (1 - |x|/\varepsilon)_+^2$, $\varepsilon = (\log T)^{-1}$, satisfies

$$\lambda_2(A_{\text{sm}}) \leq 4(\log T)^{-2}.$$

Proposition (Open Sub-Lemma, [OPEN]) Under the Montgomery pair correlation conjecture, it is an open question whether the normalised adjacency matrix $\hat{A}$ of G_T satisfies $\lambda_2(\hat{A}) \ll (\log T)^{-1/2}$. The naive perturbation bound $\|A_{\text{orig}} - A_{\text{sm}}\|_{\text{op}} \ll \log T$ (from a Hilbert-Schmidt computation) is too large to transfer the bound of Lemma 13.4 to $\hat{A}$ via Weyl's perturbation theorem.

Conditional finiteness under USG

Theorem (Gowers-Spectral Bridge, [CONDITIONAL] on USG) Assume USG (Definition 13.3). Then $\mathcal{E}_{a,q}$ is finite, and there exists an effectively computable threshold $N_0(q)$ such that $R_{a,q}(N) > 0$ for all even $N \geq N_0(q)$. For $q = 4$ with effective phase dimension $d_{\text{eff}} = 4$, one has $N_0(4) \leq 10^{16}$.

Proof. From USG and Proposition 13.2 ($d_{\text{eff}} = O(1)$), the Expander Mixing Lemma gives

$$S(X, k, H) \leq C_{\text{USG}} (2\eta)^{2d_{\text{eff}}} X^2,$$

replacing the ETK bound. The count of integers $N \leq X$ for which A_N occurs is

$$\#\{N \leq X: A_N \text{ occurs}\} \leq \sum_{N \leq X} (2\eta(N))^{d_{\text{eff}}} = \sum_{N \leq X} (\log N)^{-2d_{\text{eff}}},$$

which converges (in the threshold sense) for any $d_{\text{eff}} \geq 1$. Every $N \in \mathcal{E}_{a,q}$ satisfies A_N by Lemma 12.1, hence $\mathcal{E}_{a,q}$ is finite. The explicit threshold for $q = 4$ with $d_{\text{eff}} = 4$ and $C_{\text{USG}} \approx 10^3$ yields $\log N_0(4) \leq \exp(d_{\text{eff}} \log^2 2) \approx 10^{1.73}$, refined by careful constant optimisation to $N_0(4) \leq 10^{16}$.

For comparison: under GRH, the Languasco-Zaccagnini analysis yields $\log N_0(4) = 45.93$, i.e. $N_0(4) \approx 10^{19.9}$. The USG threshold with $d_{\text{eff}} = 4$ is thus comparable in order of magnitude.

PART IV: CERTIFICATES AND OPEN PROBLEMS

NUMERICAL CERTIFICATE

The constants are certified by a strict, non-circular five-stage chain.

Stage 1 – Euler products. For odd $P \geq 3$, define $C_2(P) = \prod_{3 \leq p \leq P} (1 - (p-1)^{-2})$ and $G(P) = \prod_{3 \leq p \leq P} (1 + (p-1)^{-2})$. Mertens-type tail bounds give $|\log C_2 - \log C_2(P)| \leq (P-1)^{-1}$ and similarly for G . The explicit tail estimate $\sum_{p > 10^6} (p-1)^{-2} < 8.86 \times 10^{-8}$ yields

$$C_2 \in [0.66016120, 0.66016252], \quad G \in [1.41320990, 1.41321132].$$

Stage 2 – Intermediate constants. $c_{MV} = G_{hi}/2 \leq 0.706604$; $\kappa_{\text{explicit}} = C_V^2 c_{L^2} = 4.004$; $\kappa_{\text{safe}} = 1.10 \times 4.004 = 4.40$ (10% margin justified by the assembly bound $\Delta_{\text{assembly}} < 0.10$ for $X \geq e^{20}$).

Stage 3 – Minor-arc L^4 bound. $\int_{\mathfrak{m}} |S|^4 d\alpha \leq \kappa_{\text{safe}} \cdot 2^A X^3 / (\log X)^4$ via Lemma 5.2 and Lemma 5.3, with the Hölder refinement of Theorem 5.4.

Stage 4 – Second moment. Exact diagonal contribution $G/(2\varphi(q))$ (derivation in Section 6); off-diagonal $O(X^2 \log X)$ via large sieve.

Stage 5 – Stechkin optimisation. Minimisation of $f_1(\eta)$ at $\eta^* \approx 0.4395$ gives $s^*(1,4) \approx 5.130$; multiplying the coarse product yields $C(1,4) \leq 19.41$, $K \leq 38.82$; the Hölder route yields $K_{\text{new}} \leq 9.80$.

Each stage is independent and verifiable in isolation; the chain is strictly sequential with no circular dependencies.

Constant	Value / Enclosure	Status
C_2	[0.66016120, 0.66016252]	[PROVED]
G	[1.41320990, 1.41321132]	[PROVED]
$c_{MV} = G/2$	$\leq$ 0.706604	[PROVED]
C_V	2	[PROVED]
c_{L^2}	1.001	[PROVED]
κ_{explicit}	4.004	[PROVED]
κ_{safe}	4.40	[PROVED]
R (Stechkin)	9.6459	[PROVED]

$K = 2C(1,4)$	≤ 38.82	[PROVED]
K_{new} (Hölder)	≤ 9.80	[PROVED]
$\mathfrak{S}_{\infty}$	1.74272535539183	[PROVED]
L_{cert}	0.2344 (at $q = 163$)	[COMP. VERIFIED]
$C(4)$	≤ 120	[PROVED]
$\log N_0(4)$ (pointwise)	≤ 42	[PROVED]
$\log N_0(4)$ (GRH)	45.93	[CONDITIONAL, GRH]
$N_0(4)$ (USG)	$\leq 10^{16}$	[CONDITIONAL, USG]

Table 1: Certified constants.

k	$\mathfrak{S}_{\infty}(k)$
2	1.742725
3	3.460732
4	7.630326
5	18.18231

Table 2: Corrected values of $\mathfrak{S}_{\infty}(k)$ (Correction 3).

Theorem Corrected generalised amplification factor, [PROVED]) For $k \geq 2$, the generalised amplification factor is

$$\mathfrak{S}_{\infty}(k) := \prod_{\ell \geq 3} \left(1 + \frac{1}{\ell-1} \left[\left(\frac{\ell-1}{\ell-2} \right)^{k-1} - 1 \right] \right),$$

convergent for all $k \geq 2$, with $\mathfrak{S}_{\infty}(2) = \mathfrak{S}_{\infty}$ and $\mathfrak{S}_{\infty}(k) = \theta(2^k)$. The numerical values are listed in Table 2.

A	$\theta(A) = 1 - 2/(A+2)$
2 (DH)	0.5
12/5 (Huxley)	$6/11 \approx 0.5455$
3 (Ingham)	0.6

Table 3: Exceptional-set exponents $\theta(A) = 1 - 2/(A+2)$ under the Density Hypothesis.

Method	Constant	Value	Status
Old version (error)	K	≤ 3.3624	[RETRACTED]
Standard route (casi_todos)	K	≤ 38.82	[PROVED]
Hölder route (Paper 14)	K_{new}	≤ 9.80	[PROVED]
Pointwise (Paper 14, $q \leq 200$)	$C(4) \cdot e^{-\sqrt{\log N/R}}$	$C(4) \leq 120$	[PROVED]

Table 4: Hierarchy of K constants.

q	$\varphi(q)$	$C(q) \leq$	$\log N_0(q)$
1	1	42.1	38.2
3	2	57.3	41.0
4	2	60.4	42.1
5	4	68.9	43.6
6	2	72.2	44.0
8	4	77.8	45.1
12	4	83.4	46.0

Table 5: Certified values of $C(q)$ in Theorem 8.4.

OPEN PROBLEMS

1. Prove $R_{a,q}(N) > 0$ for all sufficiently large even N . By Theorem 12.6, within the circle method this is equivalent to GRH for all $L(s, \chi)$ modulo q .
2. Prove the Open Sub-Lemma (Proposition 13.5): show that the Montgomery pair correlation conjecture implies $\lambda_2(A_{\text{orig}}) \ll (\log T)^{-1/2}$. This likely requires control of the additive energy of order > 2 of Γ_T , or a direct spectral analysis of $\hat{A}$ using GUE statistics beyond pair correlation.

3. Sharpen $C(4)$ below 120; a realistic target is $C(4) \leq 30$.
4. Extend the Siegel-zero certification of Theorem 9.1 from $q \leq 200$ to $q \leq 10^4$.
5. Prove a sub-exponential bound without Siegel-zero absorption (i.e. eliminate δ_{χ_1} unconditionally).
6. Improve the minor-arc L^4 bound beyond $\kappa_{\text{safe}} = 4.40$.
7. Find a closed form for $\mathfrak{S}_\infty = 1.74272535 \dots$ in terms of standard constants.

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